

Analyzing the cosmic variance limit of remote dipole measurements of the cosmic microwave background using the large-scale kinetic Sunyaev Zel'dovich effect

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Due to cosmic variance we cannot learn any more about large-scale inhomogeneities from the primary cosmic microwave background (CMB) alone. More information on large scales is essential for resolving large angular scale anomalies in the CMB. Here we consider cross correlating the large-scale kinetic Sunyaev Zel'dovich (kSZ) effect and probes of large-scale structure, a technique known as kSZ tomography. The statistically anisotropic component of the cross correlation encodes the CMB dipole as seen by free electrons throughout the observable Universe, providing information about long wavelength inhomogeneities. We compute the large angular scale power asymmetry, constructing the appropriate transfer functions, and estimate the cosmic variance limited signal to noise for a variety of redshift bin configurations. The signal to noise is significant over a large range of power multipoles and numbers of bins. We present a simple mode counting argument indicating that kSZ tomography can be used to estimate more modes than the primary CMB on comparable scales. This paper motivates a more systematic investigation of how close to the cosmic variance limit it will be possible to get with future observations.

I. INTRODUCTION

The cosmic microwave background (CMB) has been an extraordinarily powerful tool for precision cosmology, establishing the standard model, Λ CDM, at high confidence. However, on very large-scales, CMB measurements are limited by cosmic variance, implying that we can not hope to learn any more about large scale inhomogeneities from the primary CMB alone. Given that nearly all of the hints we have of departures from Λ CDM are on the very largest scales (for a recent summary of CMB anomalies see [1]), there is strong motivation to go beyond the primary CMB to learn more. Constraints from probes of Large Scale Structure (LSS), such as next-generation galaxy surveys (e.g [2]) and 21cm measurements (e.g. [3, 4]), are poised to become increasingly important for many cosmological parameters. Even still, when it comes to measuring inhomogeneities on scales \gtrsim Gpc, there will be limited additional constraining power in all but the most ambitious scenarios (e.g. 21cm dark ages cosmology [5]). In this paper, we investigate the viability of an additional probe of large scale inhomogeneities: large scale kinetic Sunyaev Zel'dovich (kSZ) tomography.

The kSZ effect is a CMB temperature anisotropy arising from the Compton scattering of CMB photons by the bulk motion of free electrons with respect to the CMB rest frame [6]. There are two main contributions to the kSZ effect: the early-time kSZ effect which arises due to patchy reionization [7–9] (see e.g. [10] for a recent update), and the late-time kSZ effect arising from free electrons in large scale structure (e.g. galaxy clusters) and the intergalactic medium. Thus far, only the latter contribution has been detected. This was achieved by looking for the contribution to CMB temperature anisotropies induced by the pairwise motion of clusters [11–14]; a (somewhat low significance) detection has also been made at the level of the temperature angular power spectrum [15]. Next-generation “Stage 3” and “Stage 4” CMB experiments [16, 17] will have the ability to make high-significance measurements of the kSZ effect¹. Realizing the full potential of kSZ measurements will continue to rely heavily on cross correlations with future probes of LSS, making the dramatic improvements to come with the next generation of redshift surveys and 21cm measurements equally important. Such cross correlations also open the door to determining the contribution to the global kSZ signal from different redshifts, a technique known as kSZ tomography [19–26].

The science case for precision measurements of the kSZ effect is quite broad. In addition to revolutionizing our understanding of reionization, it has the power to probe missing baryons e.g. [19, 27, 28], make precision tests of gravity [29–31], probe anomalous bulk flows [32–38], constrain the properties of dark energy and dark matter [39, 40],

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¹ It will at this point become important to separate the early-time and late-time kSZ effects to maximize the science return; for a proposal in this direction see Ref. [18].

constrain the masses of neutrinos [41], test the Copernican principle [21, 42, 43], constrain the present day vacuum decay rate [44], and test the hypothesis that we inhabit an eternally inflating multiverse [45].

The contribution to the kSZ effect from each free electron is proportional to the locally observed CMB dipole, and because each free electron probes a different portion of the surface of last scattering, measurements of the kSZ effect can in principle yield information about the homogeneity of the Universe. This is why measurements of the kSZ effect can be so constraining for scenarios that predict a deviation from large-scale homogeneity, such as many of those listed above. Note that this is a dramatically different regime than the one typically explored, for example in the pairwise motion of clusters that yielded the first detection. This large-scale kSZ effect is sensitive to the Sachs Wolfe and integrated Sachs Wolfe components of the dipole, in addition to the Doppler component from peculiar velocities, and can therefore in principle yield more large-scale information than peculiar velocity surveys or direct measurements of the density field through various tracers of LSS. As mentioned above, cross correlation with tracers of LSS, and therefore kSZ tomography, is key to extracting the most information possible. As we show in more detail below, information about large scale homogeneity is encoded in a statistical anisotropy of the direct correlation of tracers of large scale structure and the small angular scale CMB, e.g. a power asymmetry. Importantly, the contribution to this signal from small-scale peculiar velocities vanishes [20].

The power of kSZ tomography to probe the large scale homogeneity of the Universe has been highlighted previously, notably in Refs. [21, 33, 44, 45]². These papers considered theoretical extensions to Λ CDM where a signal could hopefully be detected with current and near-future experiments. However, as the sensitivity and resolution of CMB experiments continues to develop and as our ability to probe LSS improves, we might hope to enter an era where large scale kSZ tomography becomes a tool not just for constraining exotic scenarios, but for measuring the inhomogeneities we know to exist: those responsible for the large scale temperature anisotropies in the primary CMB.

The goal of this paper is to explore this eventuality in the most optimistic, cosmic variance limited, scenario. More specifically, we compute the angular spectrum of the asymmetry in the kSZ-LSS cross power expected in Λ CDM as a function of redshift. Comparing this signal to the accidental power asymmetry expected from the statistically isotropic components of the kSZ effect (the dominant source of CMB temperature anisotropies on small angular scales), we find that the signal-to-noise can be significant ($S/N \sim \mathcal{O}(10^2 - 10^3)$) over a wide range of angular scales ($\ell_{\max} \sim \mathcal{O}(100)$) and in a large number ($N_{\text{bin}} \sim \mathcal{O}(10 - 100)$) of redshift bins. A simple mode counting argument indicates that there is in principle more information in the power asymmetry than in the primary CMB on the relevant scales, for a sufficient number of redshift bins ($N_{\text{bin}} \gtrsim 30$).

The plan of the paper is as follows. In Sec. II, the large-scale late-time kSZ effect is summarized along with a derivation of the large-scale effective velocity. Section III describes simulations of the large-scale effective velocity field. Section IV outlines how kSZ tomography can be used to extract the large-scale effective velocity. Then, we derive the cosmic variance limited noise in Sec. V and estimate the signal using both an RMS estimate and simulations in Sec. VI. Finally, in Sec. VII we provide an estimate for the number of modes that can be obtained using cosmic variance limited kSZ tomography, showing that in principle more information can be extracted than is contained in the primary CMB on comparable scales; we conclude in Sec. VIII. A number of results are collected in the Appendix.

II. THE LARGE-SCALE KSZ EFFECT

The kinetic Sunyaev Zel'dovich effect arises from Compton scattering of CMB photons by free electrons moving with respect to the CMB rest frame. This produces temperature anisotropies given by an integral along the line of sight:

$$\left. \frac{\Delta T}{T} \right|_{\text{kSZ}}(\hat{\mathbf{n}}_e) = -\sigma_T \int_0^{\chi_{\text{re}}} d\chi_e a_e(\chi_e) n_e(\hat{\mathbf{n}}_e, \chi_e) \mathbf{v}_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) \cdot \hat{\mathbf{n}}_e \quad (1)$$

$$= -\sigma_T \int_0^{\chi_{\text{re}}} d\chi_e a_e(\chi_e) \bar{n}_e(\chi_e) (1 + \delta(\hat{\mathbf{n}}_e, \chi_e)) v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) . \quad (2)$$

The geometry is depicted in Fig. 1. In Eq. (1), σ_T is the Thomson cross-section, $n_e(\hat{\mathbf{n}}_e, \chi_e)$ is the electron number density, $\hat{\mathbf{n}}_e$ is the angular direction on the sky to the scatterer, and χ_e is the comoving radial coordinate to the

² The polarized component of the late-time kSZ effect also has the potential to constrain homogeneity on large scales [46–52], but we do not consider it further here.

scatterer along our past light cone,

$$\chi_e = \int_0^{z_e} \frac{dz}{H(z)} = - \int_1^{a_e} \frac{da}{H(a)a^2} , \quad (3)$$

where z_e and a_e are the scatterer's redshift and scale factor respectively. Below, we will use χ_e and z_e interchangeably. In the second line of (2), we have written the electron number density as $n_e(\hat{\mathbf{n}}_e, \chi_e) = \bar{n}_e(\chi_e)(1 + \delta(\hat{\mathbf{n}}_e, \chi_e))$ in terms of the average electron number density $\bar{n}_e(\chi_e)$, and the density perturbation δ , and replaced $\mathbf{v}_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) \cdot \hat{\mathbf{n}}_e$ with the projection along the line of sight $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e)$.

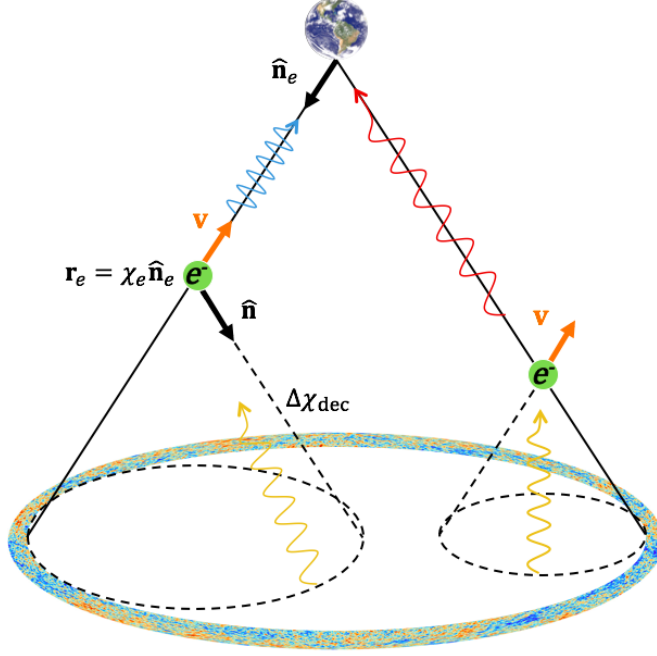


FIG. 1: Scattering of CMB photons off free electrons on our past light cone. The position of an electron is described in terms of its direction $\hat{\mathbf{n}}_e$ and comoving distance χ_e . The direction from the electron to a point on the surface of last scattering is denoted by $\hat{\mathbf{n}}$ and the distance to last scattering by $\Delta\chi_{\text{dec}}$.

The quantity denoted by $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e)$ is the CMB dipole observed by each electron, projected along the line of sight:

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = \frac{3}{4\pi} \int d^2\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) , \quad (4)$$

where, for a freely falling electron at position $\mathbf{r}_e \equiv \chi_e \hat{\mathbf{n}}_e$, the CMB temperature it sees along the direction $\hat{\mathbf{n}}$ is given by

$$\Theta(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) = \Theta_{\text{SW}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) + \Theta_{\text{Doppler}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) + \Theta_{\text{ISW}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) . \quad (5)$$

The three contributions come from the Sachs-Wolfe (SW) effect generated by the gravitational potential on the LSS, the Doppler effect due to peculiar motion of photons on the LSS and peculiar motion of electrons at redshift z_e , and the integrated Sachs-Wolfe (ISW) effect.

Working in Newtonian gauge

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 - 2\Psi)d\mathbf{x}^2 , \quad (6)$$

the Sachs-Wolfe contribution is given by

$$\Theta_{\text{SW}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) = \left(2D_{\Psi}(\chi_{\text{dec}}) - \frac{3}{2} \right) \Psi_i(\mathbf{r}_{\text{dec}}) , \quad (7)$$

where $\mathbf{r}_{\text{dec}} \equiv \chi_e \hat{\mathbf{n}}_e + \Delta\chi_{\text{dec}} \hat{\mathbf{n}}$ with $\Delta\chi_{\text{dec}} = \Delta\chi(a_{\text{dec}}) = - \int_{a_e}^{a_{\text{dec}}} da (H(a)a^2)^{-1}$ the distance along the electron's past light cone to decoupling. More generally, we will define

$$\Delta\chi(a) = - \int_{a_e}^a \frac{da}{H(a)a^2} . \quad (8)$$

In Eq. (7) we have used the growth function, $D_\Psi(a)$, which relates the potential to its primordial value at $a \rightarrow 0$ through the definition $\Psi(\mathbf{r}, a) = D_\Psi(a)\Psi_i(\mathbf{r})$. The growth function is well approximated on superhorizon scales by

$$D_\Psi(a) \equiv \frac{\Psi_{\text{SH}}(a)}{\Psi_{\text{SH},i}} = \frac{16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16}{10y^3} \left[\frac{5}{2} \Omega_m \frac{E(a)}{a} \int_0^a \frac{da}{E^3(a) a^3} \right], \quad (9)$$

where $E(a) = \sqrt{\Omega_m a^{-3} + \Omega_\Lambda}$ is the normalized Hubble parameter.

The Doppler component is given by

$$\Theta_{\text{Doppler}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot [\mathbf{v}(\mathbf{r}_e, \chi_e) - \mathbf{v}(\mathbf{r}_{\text{dec}}, \chi_{\text{dec}})] . \quad (10)$$

The velocities can be related to the potential through

$$\mathbf{v} = -\frac{2a^2 c^2 H(a)}{H_0^2 \Omega_m} \frac{y}{4+3y} \left[\nabla \Psi + \frac{d\nabla \Psi}{d \ln a} \right], \quad (11)$$

which is valid on all scales. On large scales, we can use this expression to define a velocity growth function $D_v(a)$:

$$\mathbf{v} = -\frac{2a^2 c^2 H(a)}{H_0^2 \Omega_m} \frac{y}{4+3y} \left[D_\Psi + \frac{dD_\Psi}{d \ln a} \right] \nabla \Psi_i, \quad (12)$$

where

$$D_v(a) \equiv \frac{2a^2 H(a)}{H_0^2 \Omega_m} \frac{y}{4+3y} \left[D_\Psi + \frac{dD_\Psi}{d \ln a} \right]. \quad (13)$$

Finally, the ISW term is given by

$$\Theta_{\text{ISW}}(\hat{\mathbf{n}}_e, \chi_e, \hat{\mathbf{n}}) = 2 \int_{a_{\text{dec}}}^{a_e} \frac{d\Psi}{da}(\mathbf{r}(a), a) da = 2 \int_{a_{\text{dec}}}^{a_e} \frac{dD_\Psi}{da} \Psi_i(\mathbf{r}(a)) da. \quad (14)$$

Here, $\mathbf{r}(a) = \mathbf{r}_e + \Delta\chi(a) \hat{\mathbf{n}}$.

A. Fourier kernel for the effective velocity

Relating each contribution to the effective velocity to the primordial potential Ψ_i allows us to define a kernel in Fourier space relating Ψ_i to the effective velocity. The details can be found in Appendix A, which results in the expression

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = i \int \frac{d^3 k}{(2\pi)^3} T(k) \tilde{\Psi}_i(\mathbf{k}) \mathcal{K}^v(k, \chi_e) \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e}, \quad (15)$$

where we have incorporated the transfer function $T(k)$ for the potential to account for sub-horizon evolution on small scales and the kernel $\mathcal{K}^v(k, \chi_e)$ receives contributions from the SW, ISW, and Doppler terms

$$\mathcal{K}^v(k, \chi_e) \equiv [\mathcal{K}_D(k, \chi_e) + \mathcal{K}_{\text{SW}}(k, \chi_e) + \mathcal{K}_{\text{ISW}}(k, \chi_e)], \quad (16)$$

given by

$$\mathcal{K}_D(k, \chi_e) \equiv k D_v(\chi_{\text{dec}}) j_0(k \Delta\chi_{\text{dec}}) - 2k D_v(\chi_{\text{dec}}) j_2(k \Delta\chi_{\text{dec}}) - k D_v(\chi_e) \quad (17)$$

$$\mathcal{K}_{\text{SW}}(k, \chi_e) \equiv 3 \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) j_1(k \Delta\chi_{\text{dec}}) \quad (18)$$

$$\mathcal{K}_{\text{ISW}}(k, \chi_e) \equiv 6 \int_{a_{\text{dec}}}^{a_e} da \frac{dD_\Psi}{da} j_1(k \Delta\chi(a)) \quad (19)$$

The SW and ISW kernels have support predominantly on large scales, while the Doppler kernel has support on all scales. The last term in the Doppler kernel, $k D_v(\chi_e)$, yields the “conventional” kSZ effect, and represents the dominant contribution to the kSZ effect on scales that have currently been measured (e.g. using pairwise cluster velocities). Using the fact that $j_1(k \Delta\chi(a))$ and $j_2(k \Delta\chi(a))$ have support predominantly on scales $k \sim 1/\Delta\chi$, we can

estimate the order of magnitude of scales that contribute to these terms. In the range of redshift between $6 > z_e > 0$, we have $1.25 < H_0 \Delta\chi_{\text{dec}} < 3.18$. Using $k = 1/\Delta\chi_{\text{dec}}$, this translates into the range $(14.3 \text{ Gpc})^{-1} < k < (5.6 \text{ Gpc})^{-1}$.

There is one important physical condition that must hold: a pure potential gradient should not contribute to an observable like the kSZ effect [53]. A pure gradient can be removed in linear perturbation theory by performing a special conformal transformation on the spatial metric. More generally, it is always possible to remove the gradient at a point by the same special conformal transformation. We include a proof of these statements in Appendix B. This absence of a gradient contribution to the kSZ effect has important implications for the behavior of the effective velocity Fourier kernel \mathcal{K}^v at small k .

To see this, consider a Newtonian potential that is a pure gradient

$$\Psi_i(\mathbf{x}) = A_j x^j. \quad (20)$$

Using the properties of the derivative of the Dirac delta function, we can write this in Fourier space as

$$\Psi_i(\mathbf{k}) = i(2\pi)^3 A^j \partial_j \delta^3(\mathbf{k}). \quad (21)$$

Evaluating Eq. (15), we obtain

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = -A^j \partial_j \mathcal{K}^v(k=0, \chi_e), \quad (22)$$

where we have used the fact that $\mathcal{K}^v(k=0, \chi_e) = 0$, $T(k=0) = 1$, and $\partial_j T(k=0) = 0$. Unless $\partial_j \mathcal{K}^v(k=0, \chi_e) = 0$, there will be an observable kSZ effect from a pure gradient, which would be unphysical. This, together with the fact that each of the three contributions to $\mathcal{K}^v(k, \chi_e)$ are odd functions of k , implies that we must have

$$\mathcal{K}^v(k \rightarrow 0, \chi_e) = \mathcal{O}(k^3) + \dots \quad (23)$$

Expanding \mathcal{K}_{SW} , \mathcal{K}_{ISW} , and \mathcal{K}_{D} separately, the leading order term in the Taylor series expansion is linear in k . Therefore, a cancellation between these terms must occur in the limit $k \rightarrow 0$. This is the same type of cancellation demonstrated for the primary CMB dipole due to a pure gradient in Ref. [53]. We demonstrate this cancellation analytically in a Universe with matter and a cosmological constant in Appendix C. The cancellation, and the behavior of the full kernel $\mathcal{K}^v(k, \chi_e)$ at $z = 1$ for small k is shown in Fig. 2.

Looking at the inset of the right panel in Fig. 2, we see that the contributions from the spherical Bessel functions in the SW, ISW, and Doppler kernels are evident as oscillations on the linear Doppler contribution. These effects are significant on scales up to of order $k \sim 10 H_0 \sim 2 \times 10^{-3} \text{ Mpc}^{-1}$.

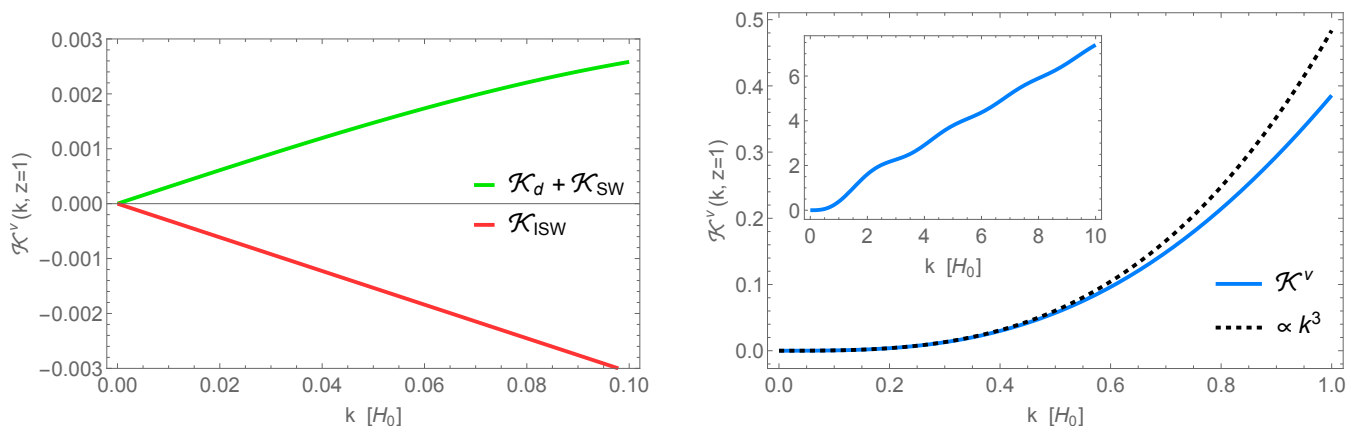


FIG. 2: The cancellation at linear order of the three pieces of \mathcal{K}^v as $k \rightarrow 0$ in Λ CDM. This cancellation can be shown analytically in a universe without radiation (see Appendix C). The leading order behavior is cubic as $k \rightarrow 0$ as shown in the right panel.

B. Angular decomposition of the effective velocity

The multipole moments of the effective velocity are given by integrating over $\hat{\mathbf{n}}_e$

$$a_{\ell m}^v(\chi_e) = \int d^2\hat{\mathbf{n}}_e v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) Y_{\ell m}^*(\hat{\mathbf{n}}_e) \quad (24)$$

$$= i \int \frac{d^3k}{(2\pi)^3} T(k) \tilde{\Psi}_i(\mathbf{k}) \mathcal{K}^v(k, \chi_e) \int d^2\hat{\mathbf{n}}_e Y_{\ell m}^*(\hat{\mathbf{n}}_e) \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (25)$$

The second integral can be written as a triple product of spherical harmonics by expanding the exponential using (A4) and writing the Legendre polynomials in terms of spherical harmonics using

$$\mathcal{P}_\ell(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{x}}) Y_{\ell m}(\hat{\mathbf{x}}') . \quad (26)$$

The integral over $\hat{\mathbf{n}}_e$ then becomes

$$\int d^2\hat{\mathbf{n}}_e Y_{\ell m}^*(\hat{\mathbf{n}}_e) \left[\frac{4\pi}{3} \sum_{m''=-1}^1 Y_{1m''}^*(\hat{\mathbf{k}}) Y_{1m''}(\hat{\mathbf{n}}_e) \right] \left[4\pi i^{\ell'} j_{\ell'}(k\chi_e) \sum_{m'=-\ell'}^{\ell'} Y_{\ell'm'}(\hat{\mathbf{k}}) Y_{\ell'm'}^*(\hat{\mathbf{n}}_e) \right] \quad (27)$$

$$= \frac{(4\pi)^2}{3} \sum_{\ell', m', m''} i^{\ell'} j_{\ell'}(k\chi_e) Y_{1m''}^*(\hat{\mathbf{k}}) Y_{\ell'm'}(\hat{\mathbf{k}}) \int d^2\hat{\mathbf{n}}_e Y_{\ell m}^*(\hat{\mathbf{n}}_e) Y_{\ell'm'}^*(\hat{\mathbf{n}}_e) Y_{1m''}(\hat{\mathbf{n}}_e) . \quad (28)$$

The triple product integral can be expressed in terms of the Clebsch-Gordan coefficients $C_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3}$:

$$\int d^2\hat{\mathbf{n}}_e Y_{\ell m}^*(\hat{\mathbf{n}}_e) Y_{\ell'm'}^*(\hat{\mathbf{n}}_e) Y_{1m''}(\hat{\mathbf{n}}_e) = \sqrt{\frac{(2\ell+1)(2\ell'+1)}{12\pi}} C_{000}^{\ell\ell'1} C_{mm'm''}^{\ell\ell'1} . \quad (29)$$

These first coefficients $C_{000}^{\ell\ell'1}$ are only nonzero for $\ell' = \ell \pm 1$. The second coefficients $C_{mm'm''}^{\ell\ell'1}$ then require that $m' = m'' - m$ for $m'' = -1, 0, 1$. Therefore, the sums over ℓ' and m' in (28) will select six non-zero terms in which (ℓ', m') take the values $(\ell+1, 1-m)$, $(\ell+1, -m)$, $(\ell+1, -1-m)$, $(\ell-1, 1-m)$, $(\ell-1, -m)$, $(\ell-1, -1-m)$. Further, spherical harmonic identities and spherical Bessel recursion relations can then be used to simplify these six terms into just two terms proportional to $Y_{\ell m}^*(\hat{\mathbf{k}})$. Equation (28) reduces to

$$\frac{4\pi}{2\ell+1} Y_{\ell m}^*(\hat{\mathbf{k}}) [i^{\ell-1} \ell j_{\ell-1}(k\chi_e) + i^{\ell+1} (\ell+1) j_{\ell+1}(k\chi_e)] . \quad (30)$$

Plugging this in for the second integral in equation (25) leads to the expression

$$a_{\ell m}^v(\chi_e) = \int \frac{d^3k}{(2\pi)^3} \Delta_\ell^v(k, \chi_e) \tilde{\Psi}_i(\mathbf{k}) Y_{\ell m}^*(\hat{\mathbf{k}}) , \quad (31)$$

where we have defined the transfer function $\Delta_\ell^v(k, \chi_e)$ as

$$\Delta_\ell^v(k, \chi_e) \equiv \frac{4\pi i^\ell}{2\ell+1} \mathcal{K}^v(k, z\chi_e) [\ell j_{\ell-1}(k\chi_e) - (\ell+1) j_{\ell+1}(k\chi_e)] T(k) . \quad (32)$$

The asymptotic behavior of the transfer function as $k \rightarrow \infty$ and $k \rightarrow 0$ is given by:

$$\lim_{k \rightarrow \infty} \Delta_\ell^v(k, \chi_e) = -4\pi i^\ell \frac{D_v(\chi_e)}{\chi_e} T(k) \cos[k\chi_e - \ell\pi/2] \quad (33)$$

$$\lim_{k \rightarrow 0} \Delta_\ell^v(k, \chi_e) = \frac{4\pi i^\ell}{2\ell+1} \left[\frac{\ell \sqrt{\pi} (k\chi_e)^{\ell-1}}{2^\ell \Gamma[\frac{1}{2} + \ell]} \right] c_3(\chi_e) k^3 , \quad (34)$$

where in the small- k limit we have used the fact that $T(0) = 1$ and written the coefficient of the leading order (cubic) term in the Taylor series expansion of $\mathcal{K}^v(k, z\chi_e)$ as $c_3(\chi_e)$; in the large- k limit, $T(k) \propto k^{-2}$.

III. SIMULATIONS

In this section we describe a suite of simulations used to explore the large-scale kSZ effect and provide a concrete example of the relation between the primordial gravitational potential and the effective velocity. This will be used to compute the kSZ signal-to-noise in section VIB. We create three dimensional realizations of the primordial gravitational potential $\Psi(\mathbf{x})$ consistent with Λ CDM using the method described in Ref. [54] and reviewed in Appendix D. The box size used in each case was $L = 7H_0^{-1} \simeq 31.3$ Gpc. One hundred realizations were created at a resolution of 128^3 , covering scales down to $k_{\text{max}} \simeq 57.4 H_0$ ($\lambda_{\text{min}} \simeq 484$ Mpc). An example realization is shown in Fig. 3.

With a set of realizations in hand, we then place a hypothetical observer at the center of the box and generate $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e)$ at 50 equally spaced values of χ_e at a Healpix resolution [55] of $N_{\text{side}} = 32$ (12,288 equal area pixels of approximately 3.36 square degrees each). This is done as follows. First, we write the effective velocity as

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = i\hat{\mathbf{n}}_e \cdot \mathbf{V}(\hat{\mathbf{n}}_e, \chi_e) \quad (35)$$

where

$$\mathbf{V}(\hat{\mathbf{n}}_e, \chi_e) \equiv \int \frac{d^3k}{(2\pi)^3} \left[T(k) \tilde{\Psi}_i(\mathbf{k}) \mathcal{K}^v(k, \chi_e) \frac{\mathbf{k}}{k} \right] e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \quad (36)$$

The three components of $\mathbf{V}(\hat{\mathbf{n}}_e, \chi_e)$ can be straightforwardly evaluated for each realization, at each redshift, using an fast Fourier transform (FFT) algorithm. Plugging back into Eq. (35) and choosing $\hat{\mathbf{n}}_e \cdot \hat{\mathbf{z}} = \cos \theta$, $\hat{\mathbf{n}}_e \cdot \hat{\mathbf{x}} = \sin \theta \cos \phi$, and $\hat{\mathbf{n}}_e \cdot \hat{\mathbf{y}} = \sin \theta \sin \phi$, we interpolate the resulting $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e)$ at each χ_e onto the Healpix grid. We then take advantage of the Healpix fast spherical harmonic transform functionality to obtain $a_{\ell m}^v(\chi_e)$ at each redshift in each realization.

In the right panel of Fig. 3, we show $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e)$ at a variety of redshifts in a single realization. In the top row, we choose two nearby redshifts, where it can be seen by eye that there is a good deal of correlation between the two maps. This is because the same large-scale potential field is responsible for the effective velocities at nearby redshifts. In the bottom row, we choose fairly distant redshifts, where the correlation between the two maps is largely absent. Note also the increasing structure with redshift. This is partially due to the limited resolution of the simulation in this figure, but more physically, there is a real effect due to the redshift transfer function. Based on the smallest structures in the resolution probed in our simulation, 64 radial samples would capture all radial structures. Empirically, for all but the smallest angular scale structures, the coherence length between redshift slices is sufficiently long to justify our choice of 50 values of χ_e .

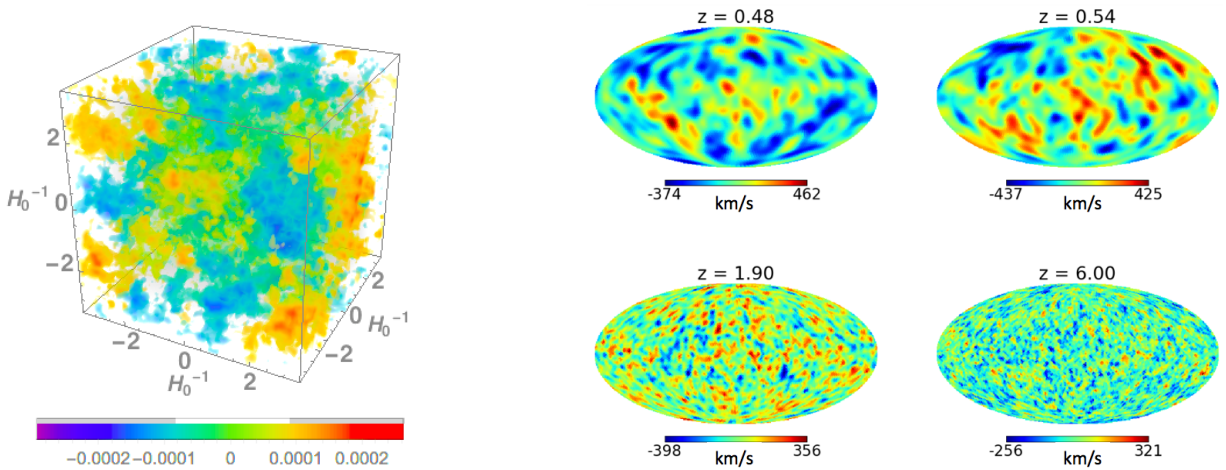


FIG. 3: By generating random realizations of $\tilde{\Psi}_i(\mathbf{k})$, drawn from a Gaussian distribution with power $P_\Psi(k)$, we can construct realizations for v_{eff} . The left panel shows an example realization for $\Psi_i(\mathbf{r})$, and the resulting maps of v_{eff} at various redshifts, using a resolution of $k_{\text{max}} \sim 57 H_0$ ($\lambda_{\text{min}} \sim 484$ Mpc). Notice that correlations are evident between the top two v_{eff} maps at close redshift, but not apparent for widely separated redshifts. Also note the increase in structure at higher redshift.

IV. KSZ TOMOGRAPHY

There is a large amount of information lost in performing the line-of-sight integral in Eq. (1) for the global kSZ signal. One can in principle do far better by cross correlating the kSZ temperature anisotropies with tracers of the electron density field of known redshift. This is evident in the first detections of the kSZ effect, which were made by isolating the component of the temperature anisotropies associated with the pairwise motion of clusters, whose hot interiors harbor a large density of free electrons. In what follows, we assume the most optimistic scenario possible, in which we have perfect knowledge of the electron density field obtained through the measurement of a completely unbiased tracer. We further assume a purely Gaussian primordial power spectrum, consistent with the current constraints from Planck [56].

To tease out the redshift dependence of the large scale kSZ effect, we introduce a window function $W(\chi_e, \bar{\chi}_e)$ that gives the electron density in a set of redshift bins centered on $\chi_e = \bar{\chi}_e$

$$\delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) = \int d\chi_e W(\chi_e, \bar{\chi}_e) \delta(\hat{\mathbf{n}}_e, \chi_e) . \quad (37)$$

In the following, we use a top-hat window function normalized to unity: $\int_0^{\chi_\infty} d\chi W(\chi, \bar{\chi}) = 1$. We will consider scenarios with 6, 12, and 24 redshift bins of equal width, covering the range $0 < z < 6$. The redshift coverage for each bin configuration is shown in Figure 4.

Forming the cross correlation between the kSZ contribution to the CMB temperature anisotropies and the windowed electron density field, we obtain

$$\left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle = \sigma_T \int d\chi'_e W(\chi'_e, \bar{\chi}_e) \int d\chi_e a(\chi_e) \bar{n}_e(\chi_e) \langle (1 + \delta(\hat{\mathbf{n}}_e, \chi_e)) v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) \delta(\hat{\mathbf{n}}'_e, \chi'_e) \rangle . \quad (38)$$

Now comes a very important step. The correlation function above is defined as an ensemble average. Typically, one is interested in using the measured correlation functions to constrain a statistical model of the ensemble. Here, this is not the case. Instead, *we strive to learn information about our particular realization* using the cross correlation, which is information that in the former scenario would have been an obstruction to learning about the theoretical model of the ensemble (e.g. cosmic variance). As we wish to learn about large-scale inhomogeneities, the ensemble average in Eq.(38) should only be taken over small scales, leaving large scales as a fixed deterministic field.

To formulate this idea more precisely, we decompose the primordial potential into long and short wavelength fields, which can be defined in Fourier space as:

$$\Psi_i^{\mathcal{L}}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \mathcal{L}(k) \Psi_i(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \Psi_i^{\mathcal{S}}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \mathcal{S}(k) \Psi_i(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (39)$$

such that $\mathcal{L}(k) + \mathcal{S}(k) = 1$. A suitable choice could be $\mathcal{L}(k) = e^{-k^2/2k_*^2}$, $\mathcal{S}(k) = 1 - e^{-k^2/2k_*^2}$, although we will implicitly be choosing a step function for $\mathcal{L}(k)$ in what follows. Below, we imagine that scales larger than $k_* \gtrsim 10^{-2} \text{ Mpc}^{-1}$ form the deterministic long field while smaller scales form the stochastic short field. The precise choice of k_* does not affect our results because as we show in Sec. V and VI small scales ($k \gg 10^{-2} \text{ Mpc}^{-1}$) do not contribute to the signal and large scales ($k \ll 10^{-2} \text{ Mpc}^{-1}$) do not contribute to the noise.

Because we are working in the linear regime, a long-short split in Ψ_i translates into a long-short split in the effective velocity v_{eff} and the electron density field δ , which we therefore decompose as:

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = v_{\text{eff}}^{\mathcal{L}}(\hat{\mathbf{n}}_e, \chi_e) + v_{\text{eff}}^{\mathcal{S}}(\hat{\mathbf{n}}_e, \chi_e), \quad \delta(\hat{\mathbf{n}}_e, \chi_e) = \delta^{\mathcal{L}}(\hat{\mathbf{n}}_e, \chi_e) + \delta^{\mathcal{S}}(\hat{\mathbf{n}}_e, \chi_e) . \quad (40)$$

Substituting these expansions into the ensemble average in Eq. (38) and extracting the long wavelength fields in the sense described above, we obtain:

$$\begin{aligned} \langle (1 + \delta) v_{\text{eff}} \delta' \rangle &= \langle (1 + \delta^{\mathcal{L}} + \delta^{\mathcal{S}}) (v_{\text{eff}}^{\mathcal{L}} + v_{\text{eff}}^{\mathcal{S}}) (\delta'^{\mathcal{L}} + \delta'^{\mathcal{S}}) \rangle \\ &= v_{\text{eff}}^{\mathcal{L}} \delta'^{\mathcal{L}} + v_{\text{eff}}^{\mathcal{L}} \delta^{\mathcal{L}} \delta'^{\mathcal{L}} \\ &\quad + v_{\text{eff}}^{\mathcal{L}} \langle \delta'^{\mathcal{S}} \rangle + \delta'^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \rangle + \delta^{\mathcal{L}} \delta'^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \rangle + v_{\text{eff}}^{\mathcal{L}} \delta^{\mathcal{L}} \langle \delta'^{\mathcal{S}} \rangle + v_{\text{eff}}^{\mathcal{L}} \delta'^{\mathcal{L}} \langle \delta^{\mathcal{S}} \rangle \\ &\quad + \langle v_{\text{eff}}^{\mathcal{S}} \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle \\ &\quad + v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle + \delta^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle + \delta'^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \delta^{\mathcal{S}} \rangle + \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle , \end{aligned} \quad (41)$$

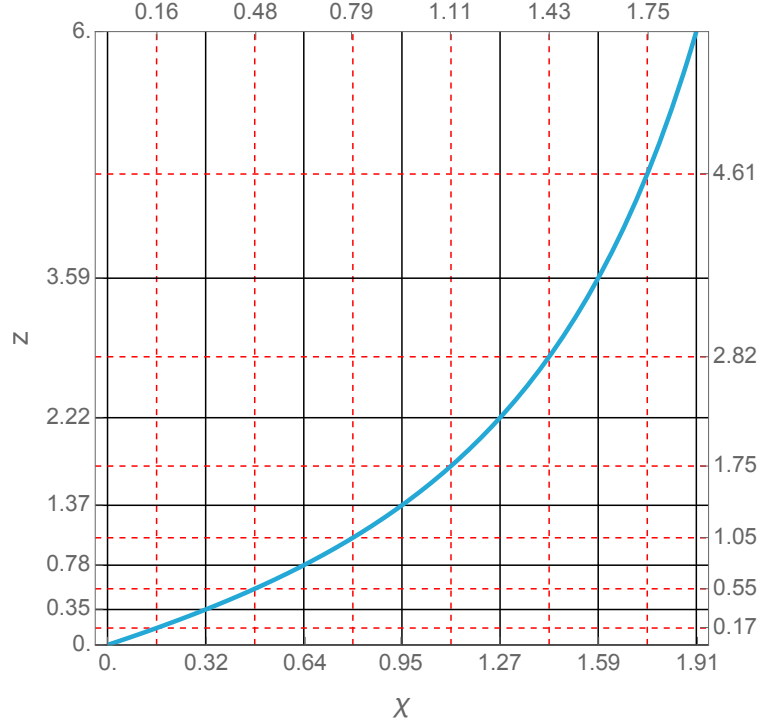


FIG. 4: Redshift is displayed as a function of comoving distance to illustrate the redshift bin configurations. The solid grid lines show the boundaries that define the redshift bins for $N_{\text{bin}} = 6$. The 12 bin configuration is represented by including the dashed grid lines. It is easy to infer the boundary values for $N_{\text{bin}} = 24$.

where δ' represents $\delta(\hat{\mathbf{n}}'_e, \chi'_e)$. From our assumption that the short wavelength components are approximately Gaussian, we set the one-point and three-point correlation functions of short wavelength fields to zero³, resulting in the final expression:

$$\begin{aligned} \langle (1 + \delta) v_{\text{eff}} \delta' \rangle &= v_{\text{eff}}^{\mathcal{L}} \delta'^{\mathcal{L}} + v_{\text{eff}}^{\mathcal{L}} \delta^{\mathcal{L}} \delta'^{\mathcal{L}} \\ &\quad + \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle \\ &\quad + v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle + \delta^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle + \delta'^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \delta^{\mathcal{S}} \rangle. \end{aligned} \quad (42)$$

The terms on the first line give rise to fluctuations on large angular scales, where the primary CMB dominates. We can therefore eliminate this hopelessly unmeasurable term by filtering the CMB on large angular scales $\ell \lesssim 3000$.⁴ The term on the second line gives rise to a statistically isotropic cross power. The terms on the third line give rise to a long wavelength modulation of small-scale power, and will be the focus of what follows. The first of these sources of power asymmetry, $v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle$, is far larger than the other two. If we consider a single long ($k^{\mathcal{L}}$) and a single short ($k^{\mathcal{S}}$) wavelength mode, then noting that the Doppler term dominates $v_{\text{eff}}^{\mathcal{S}}$ and using $v \propto \delta/k$, we have $\delta^{\mathcal{L}} \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle \sim (k^{\mathcal{L}}/k^{\mathcal{S}}) v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle \ll v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle$. We can therefore approximate

$$\langle (1 + \delta) v_{\text{eff}} \delta' \rangle \simeq \langle v_{\text{eff}}^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle + v_{\text{eff}}^{\mathcal{L}} \langle \delta^{\mathcal{S}} \delta'^{\mathcal{S}} \rangle, \quad (43)$$

illustrating that there is a statistically isotropic contribution in the first term that depends only on small scales, and an anisotropic contribution in the second term that depends on large scales. The desired signal is captured in the anisotropic power asymmetry, whereas the small scale isotropic component contributes to the noise which is computed next in Sec. V. Focusing here on the signal, substituting Eq. (43) into Eq. (38), and suppressing the \mathcal{S} and

³ In any case, significant non-Gaussianity on small scales will not directly mimic the signal we are ultimately after, which is a *long wavelength* modulation of short wavelength power.

⁴ The kSZ effect dominates the CMB temperature anisotropies on scales $\ell \gtrsim 3000$, as shown in Fig. 6.

\mathcal{L} superscripts, we obtain

$$\left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle = \sigma_T \int d\chi_e a(\chi_e) \bar{n}_e(\chi_e) v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) \int d\chi'_e W(\chi'_e, \bar{\chi}_e) \langle \delta(\hat{\mathbf{n}}_e, \chi_e) \rangle \delta(\hat{\mathbf{n}}'_e, \chi'_e) + \text{isotropic} . \quad (44)$$

Assuming the electron distribution traces the dark matter, the electron density correlation function is given by

$$\langle \delta(\hat{\mathbf{n}}_e, \chi_e) \rangle \delta(\hat{\mathbf{n}}'_e, \chi'_e) = \sum_{\ell} \frac{2\ell+1}{4\pi} C_{\ell}^{\delta\delta}(\chi_e, \chi'_e) \mathcal{P}_{\ell}(\hat{\mathbf{n}}_e \cdot \hat{\mathbf{n}}'_e) , \quad (45)$$

$$\text{with } C_{\ell}^{\delta\delta}(\chi_e, \chi'_e) = \int \frac{dk}{(2\pi)^3} \frac{k^2}{4\pi} j_{\ell}(k\chi_e) \sqrt{P_{\delta}(k, \chi_e)} 4\pi j_{\ell}(k\chi'_e) \sqrt{P_{\delta}(k, \chi'_e)} , \quad (46)$$

where $P_{\delta}(k, \chi_e)$ is the non-linear matter power spectrum [57], which was computed using the Cosmicpy package.

We now arrive at the main conclusion of this section. By isolating the statistically anisotropic term in Eq. (44), it is possible to measure v_{eff} . Because v_{eff} is related to the primordial potential Ψ_i in linear theory by Eq. (15), this potentially opens a new observational window on large scale inhomogeneities in our Universe. Given a specific model for Ψ_i , it is possible to design an optimal filter to extract the power asymmetry described above [33, 35, 45]. However, in the present context of random Gaussian fields, we quantify the power asymmetry by decomposing into power multipoles [58] that capture the anisotropic term in (43),

$$b_{LM}(\bar{\chi}_e) = \int d^2\hat{\mathbf{n}}_e Y_{LM}^*(\hat{\mathbf{n}}_e) \left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \rangle . \quad (47)$$

Next, we expand the long-wavelength effective velocity into multipoles (see Sec. II B) using $v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = \sum_{\ell, m} a_{\ell m}^v(\chi_e) Y_{\ell m}(\hat{\mathbf{n}}_e)$ where $a_{\ell m}^v$ is given by (31). This makes the angular integral in (47) easy to compute as $\int d^2\hat{\mathbf{n}}_e Y_{LM}^*(\hat{\mathbf{n}}_e) Y_{\ell m}(\hat{\mathbf{n}}_e) = \delta_{\ell L} \delta_{m M}$, resulting in the expression,

$$b_{LM}(\bar{\chi}_e) = \sum_{\ell} \frac{2\ell+1}{4\pi} \int dk \frac{2k^2}{\pi} \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) a_{LM}^v(\chi_e) \sqrt{P_{\delta}(k, \chi_e)} j_{\ell}(k\chi_e) \int d\chi'_e W(\chi'_e, \bar{\chi}_e) \sqrt{P_{\delta}(k, \chi'_e)} j_{\ell}(k\chi'_e) . \quad (48)$$

We therefore see that the power multipoles in each bin are proportional to a weighted integral of the corresponding multipole of the projected effective velocity field.

V. COSMIC VARIANCE LIMITED NOISE

In this section we estimate the cosmic variance limited noise that we expect for the power multipoles computed in the previous section. However, above, our focus was the statistically anisotropic contribution to the cross correlation, but here we are interested in the statistically isotropic contribution to temperature anisotropies, which depends predominantly on small scales.

On small angular scales, the late-time kSZ effect is the dominant source of temperature anisotropies in the CMB (see the right panel of Fig. 6 below). Under the assumption of approximate Gaussianity, and assuming a perfectly unbiased measurement of the electron density field, the primary source of noise is therefore an “accidental” power asymmetry in the cross correlation. We can estimate this through the variance in the power multipoles, which is computed as the coincident limit of the four point function between two powers of $\frac{\Delta T}{T}|_{\text{kSZ}}$ and two powers of δ . Specifically, we must compute:

$$\langle \tilde{b}_{LM}^*(\bar{\chi}_e) \tilde{b}_{LM}(\bar{\chi}_e) \rangle = \int d^2\hat{\mathbf{n}}_e d^2\hat{\mathbf{n}}'_e Y_{LM}^*(\hat{\mathbf{n}}_e) Y_{LM}(\hat{\mathbf{n}}'_e) \left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \frac{\Delta T}{T} \Big|_{\text{kSZ}} (\hat{\mathbf{n}}'_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle , \quad (49)$$

where we are using tildes on the b_{LM} ’s to indicate that these are not simply the same power multipoles as in (48). Instead, the variance here captures the chance power asymmetry that is present in the statistically isotropic contribution to $\frac{\Delta T}{T}$, which is sensitive only to small scales where the “conventional” kSZ effect dominates. The 4-point function has contributions from all possible 2-point functions and an irreducible/connected piece (which we assume to be negligibly small)

$$\left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \frac{\Delta T}{T} \Big|_{\text{kSZ}} (\hat{\mathbf{n}}'_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle = \left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \frac{\Delta T}{T} \Big|_{\text{kSZ}} (\hat{\mathbf{n}}'_e) \rangle \langle \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle \quad (50)$$

$$+ \left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle \left\langle \frac{\Delta T}{T} \right|_{\text{kSZ}} (\hat{\mathbf{n}}'_e) \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \rangle \quad (51)$$

Assuming the electron density field traces dark matter, the electron density autocorrelation function is given by

$$\langle \delta(\hat{\mathbf{n}}_e, \bar{\chi}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle = \sum_{\ell} \frac{2\ell+1}{4\pi} C_{\ell}^{\delta\delta} \mathcal{P}_{\ell}(\hat{\mathbf{n}}_e \cdot \hat{\mathbf{n}}'_e), \quad (52)$$

$$\text{with } C_{\ell}^{\delta\delta} = \int d\chi_e W(\chi_e, \bar{\chi}_e) \int d\chi'_e W(\chi'_e, \bar{\chi}_e) C_{\ell}^{\delta\delta}(\chi_e, \chi'_e) \quad (53)$$

$$= \int dk \frac{2k^2}{\pi} \int d\chi_e \sqrt{P_{\delta}(k, \chi_e)} W(\chi_e, \bar{\chi}_e) j_{\ell}(k\chi_e) \int d\chi'_e \sqrt{P_{\delta}(k, \chi'_e)} W(\chi'_e, \bar{\chi}_e) j_{\ell}(k\chi'_e) \quad (54)$$

$$\simeq \int \frac{dk}{\ell+1/2} W^2\left(\frac{\ell+1/2}{k}, \bar{\chi}_e\right) P_{\delta}\left(k, \frac{\ell+1/2}{k}\right), \quad (55)$$

where we used the expression for $C_{\ell}^{\delta\delta}(\chi_e, \chi'_e)$ from (46), and the Limber approximation [59] in the last line. The quantity $C_{\ell}^{\delta\delta}$ is shown in Fig. 5.

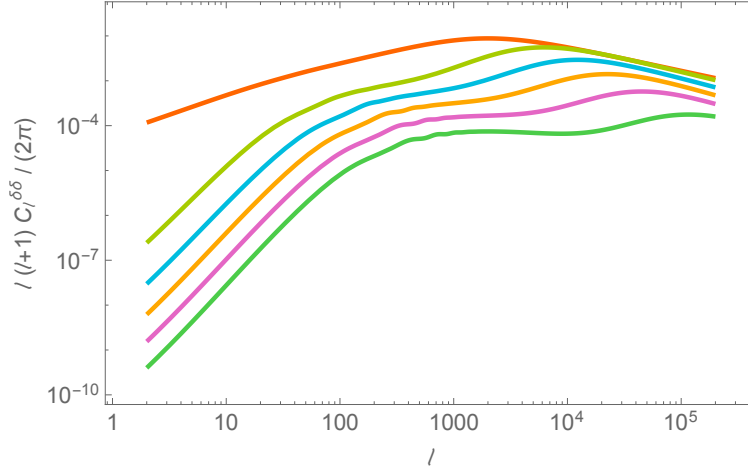


FIG. 5: The non-linear angular matter power spectrum computed in six different redshift bins described by the rows in Fig. 4 from today (top) to $z = 6$ (bottom).

The temperature autocorrelation function has contributions from the primary CMB and other secondaries. Here we consider only the primary CMB and the conventional kSZ effect, the latter dominates past $\ell \sim 3000$. The kSZ contribution has the following form:

$$\left\langle \frac{\Delta T}{T} \right\rangle_{\text{kSZ}}(\hat{\mathbf{n}}_e) \frac{\Delta T}{T} \Big|_{\text{kSZ}}(\hat{\mathbf{n}}'_e) \rangle = \int d\chi_e \sigma_{TA}(\chi_e) \bar{n}_e(\chi_e) \int d\chi'_e \sigma_{TA}(\chi'_e) \bar{n}_e(\chi'_e) \langle q(\chi_e, \hat{\mathbf{n}}_e) q(\chi'_e, \hat{\mathbf{n}}'_e) \rangle \quad (56)$$

$$= \sum_{\ell} \frac{2\ell+1}{4\pi} C_{\ell}^{TT, \text{kSZ}} \mathcal{P}_{\ell}(\hat{\mathbf{n}}_e \cdot \hat{\mathbf{n}}'_e), \quad (57)$$

where $q \equiv \mathbf{q} \cdot \hat{\mathbf{n}}_e \equiv v_{\text{eff}}(1 + \delta)$ is the momentum field of free electrons. To obtain an expression for the temperature power spectrum, $C_{\ell}^{TT, \text{kSZ}}$, the key quantity to compute is $\langle q(\chi_e, \hat{\mathbf{n}}_e) q(\chi'_e, \hat{\mathbf{n}}'_e) \rangle$, which is the fourth moment of two δ 's and two v 's. Schematically, $\langle qq \rangle = \langle vv \rangle \langle \delta\delta \rangle + 2\langle v\delta \rangle^2 + \langle v\delta v\delta \rangle_c$, where the subscript c denotes the irreducible connected term. The momentum power spectrum, denoted by P_q , is typically computed by decomposing the Fourier transform, $\tilde{\mathbf{q}}(\mathbf{k})$, into components parallel to $\hat{\mathbf{k}}$, $\tilde{\mathbf{q}}_{\parallel} = \hat{\mathbf{k}}(\tilde{\mathbf{q}} \cdot \hat{\mathbf{k}})$, and perpendicular to $\hat{\mathbf{k}}$, $\tilde{\mathbf{q}}_{\perp} = \tilde{\mathbf{q}} - \hat{\mathbf{k}}(\tilde{\mathbf{q}} \cdot \hat{\mathbf{k}})$ [60]. The longitudinal momentum component does not contribute significantly to $C_{\ell}^{TT, \text{kSZ}}$ due to cancellations of positive and negative contributions in the line-of-sight integration. For instance, as shown by Park et al. [61], the longitudinal contribution to $C_{\ell}^{TT, \text{kSZ}}$ is more than four orders of magnitude below the transverse contribution for $\ell > 3000$, so it suffices to only consider $P_{q_{\perp}}$ in this calculation.

To compute $P_{q_{\perp}}$, we use the “standard kSZ model” [61, 62], which incorporates the fully non-linear power spectrum for the density field, P_{δ}^{nl} , but approximates the velocity power spectrum by linear theory $P_v = \frac{a^2 f^2}{k^2} P_{\delta}^{\text{lin}}$ where

$f = \left(1 + \frac{a}{D_\Psi} \frac{dD_\Psi}{da}\right)^2$. The resulting expression is

$$P_{q\perp}(k, \chi) = \dot{a}^2 f^2 \int_{-1}^1 d\mu \int \frac{dk'}{(2\pi)^2} P_\delta^{\text{lin}}(k', \chi) P_\delta^{\text{nl}}(\sqrt{k^2 - 2kk'\mu + k'^2}, \chi) \frac{(k^2 - 2kk'\mu)(1 - \mu^2)}{k^2 - 2kk'\mu + k'^2}. \quad (58)$$

This model neglects the velocity-density cross correlation because the geometrical factor attached to this term decreases rapidly at large k . Since the kSZ contribution to C_ℓ^{TT} that we are interested is sensitive only to high k scales, this approximation is valid. Note that this also implies our precise choice of k_* in Sec. IV is irrelevant for the noise calculation. We are also neglecting the non-Gaussian contribution from the connected 4-point function (unimportant on all scales according to [60], but could account for up to 10% of the power spectrum according to [61]). In terms of $P_{q\perp}$ the temperature power spectrum can be expressed as

$$C_\ell^{TT, \text{kSZ}} = \int dk \frac{2k^2}{\pi} \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) \sqrt{P_{q\perp}(k, \chi_e)} j_\ell(k\chi_e) \int d\chi'_e \sigma_T a(\chi'_e) \bar{n}_e(\chi'_e) \sqrt{P_{q\perp}(k, \chi'_e)} j_\ell(k\chi'_e) \quad (59)$$

$$\sim \int \frac{dk}{\ell + 1/2} [\sigma_T a(\chi) \bar{n}_e(\chi)]^2 P_{q\perp}(k, \chi) \Big|_{\chi \rightarrow (\ell+1/2)/k}, \quad (60)$$

using the Limber approximation [59] again. The functions $P_{q\perp}(k, \chi)$ and $C_\ell^{TT, \text{kSZ}}$ are shown in Fig. 6.

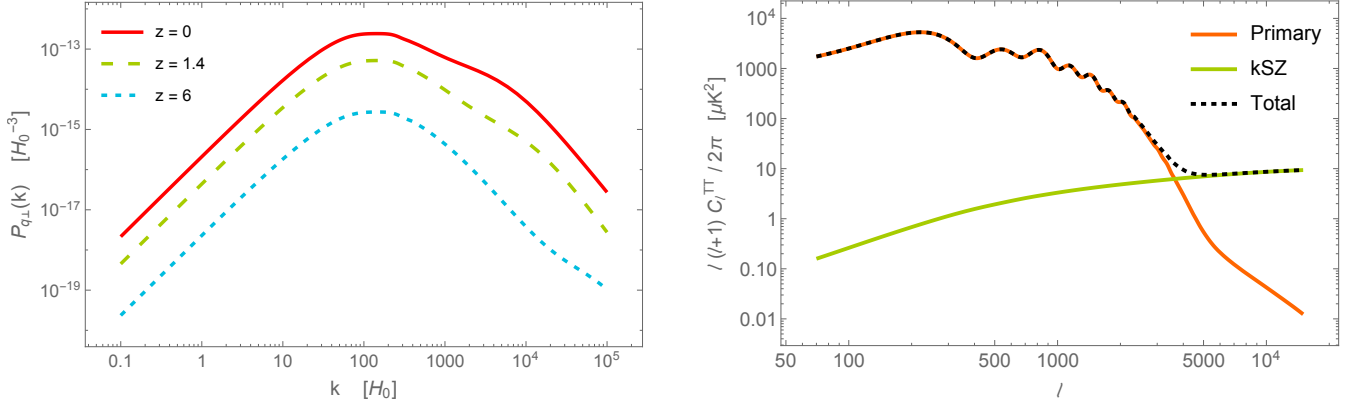


FIG. 6: Left: the power spectrum of the transverse momentum field (58).

Right: the contributions to the temperature-temperature power spectrum from the primary CMB and the non-linear kSZ effect (60).

The second term in Eq. (51) requires us to compute the cross correlation of the temperature with density. This can be done as follows

$$\left\langle \frac{\Delta T}{T} \right\rangle_{\text{kSZ}} (\hat{\mathbf{n}}_e) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) = - \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) \langle v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) (1 + \delta(\hat{\mathbf{n}}_e, \chi_e)) \delta(\hat{\mathbf{n}}'_e, \bar{\chi}_e) \rangle \quad (61)$$

$$= - \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) \int d\chi'_e W(\chi'_e, \bar{\chi}_e) \langle v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) \delta(\hat{\mathbf{n}}'_e, \chi'_e) \rangle \quad (62)$$

$$= \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell^{T\delta} \mathcal{P}_\ell(\hat{\mathbf{n}}_e \cdot \hat{\mathbf{n}}'_e). \quad (63)$$

Assuming approximate Gaussianity, we neglect the three point function $\langle v\delta\delta \rangle$, so only the correlation between v_{eff} and δ remains. Recall that we are computing the contribution from the small scales, whereas the signal calculation above (see Eqs. (43),(44)) is only sensitive to large scales. The power spectrum $C_\ell^{T\delta}$ takes the form

$$C_\ell^{T\delta} = - \int \frac{dk}{(2\pi)^3} \frac{k^2}{k} \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) \Delta_\ell^v(k, \chi_e) \sqrt{P_\Psi(k)} \int d\chi'_e 4\pi \sqrt{P_\delta(k, \chi'_e)} W(\chi'_e, \bar{\chi}_e) j_\ell(k\chi'_e) \quad (64)$$

$$= - \int \frac{dk}{(2\pi)^3} \int d\chi_e \sigma_T a(\chi_e) \bar{n}_e(\chi_e) 4\pi \mathcal{K}^v(k, \chi_e) \frac{T(k)}{k} \frac{dj_\ell(k\chi_e)}{d\chi_e} \sqrt{P_\Psi(k)} \int d\chi'_e 4\pi \sqrt{P_\delta(k, \chi'_e)} W(\chi'_e, \bar{\chi}_e) j_\ell(k\chi'_e), \quad (65)$$

where we've recalled the expression for $\Delta_\ell^v(k, \chi_e)$ from Eq. (32), and used the identity,

$$\frac{\ell j_{\ell-1}(k\chi) - (\ell+1)j_{\ell+1}(k\chi)}{2\ell+1} = \frac{1}{k} \frac{dj_\ell(k\chi)}{d\chi}. \quad (66)$$

Integrating by parts and using the Limber approximation [59] results in the expression

$$C_\ell^{T\delta} = \int dk \frac{2k^2}{\pi} \int d\chi_e \sigma_T \frac{d}{d\chi_e} [a(\chi_e) \bar{n}_e(\chi_e) \mathcal{K}^v(k, \chi_e)] \frac{T(k)}{k} \sqrt{P_\Psi(k)} j_\ell(k\chi_e) \int d\chi'_e \sqrt{P_\delta(k, \chi'_e)} W(\chi'_e, \bar{\chi}_e) j_\ell(k\chi'_e) \quad (67)$$

$$\simeq \int \frac{dk}{\ell+1/2} \sigma_T \frac{d}{d\chi} [a(\chi) \bar{n}_e(\chi) \mathcal{K}^v(k, \chi)] \frac{T(k)}{k} \sqrt{P_\Psi(k) P_\delta(k, \chi)} W(\chi, \bar{\chi}_e) \Big|_{\chi \rightarrow (\ell+1/2)/k}. \quad (68)$$

Combining all of these pieces, and expanding $\mathcal{P}_\ell(\hat{\mathbf{n}}_e \cdot \hat{\mathbf{n}}'_e)$ in terms of spherical harmonics using the identity (26), the variance in the power multipoles can then be written

$$\langle \tilde{b}_L^2 \rangle = \sum_M \frac{\langle \tilde{b}_{LM}^* \tilde{b}_{LM} \rangle}{2L+1} = \sum_{\ell, \ell', m, m', M} \frac{C_\ell^{TT, \text{kSZ}} C_{\ell'}^{\delta\delta} + C_\ell^{T\delta} C_{\ell'}^{T\delta}}{2L+1} \left| \int d^2 \hat{\mathbf{n}}_e Y_{LM}(\hat{\mathbf{n}}_e) Y_{\ell m}^*(\hat{\mathbf{n}}_e) Y_{\ell' m'}^*(\hat{\mathbf{n}}_e) \right|^2 \quad (69)$$

$$= \sum_{\ell, \ell'} [C_\ell^{TT, \text{kSZ}} C_{\ell'}^{\delta\delta} + C_\ell^{T\delta} C_{\ell'}^{T\delta}] \frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)^2} \left| C_{000}^{\ell\ell'L} \right|^2 \sum_{m, m', M} \left| C_{mm'M}^{\ell\ell'L} \right|^2 \quad (70)$$

$$= \sum_{\ell, \ell'} [C_\ell^{TT, \text{kSZ}} C_{\ell'}^{\delta\delta} + C_\ell^{T\delta} C_{\ell'}^{T\delta}] \frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)} \left| C_{000}^{\ell\ell'L} \right|^2, \quad (71)$$

where $C_{mm'M}^{\ell\ell'L}$ are the Clebsch-Gordan coefficients, and we used the triple product integral identity (see Eq. (29)) and $\sum_{m, m', M} \left| C_{mm'M}^{\ell\ell'L} \right|^2 = 2L+1$. Note that the coefficients $C_{000}^{\ell\ell'L}$ are only nonzero for $|\ell - \ell'| \leq L \leq \ell + \ell'$ and $\ell + \ell' + L$ even. Fig. 7 shows $(C_\ell^{T\delta})^2$ plotted in three different redshift bins, and compares this to $C_\ell^{TT, \text{kSZ}} C_\ell^{\delta\delta}$. It is clear that the cross term is subdominant by several orders of magnitude for all ℓ , and therefore it is sufficient to approximate $C_\ell^{TT, \text{kSZ}} C_{\ell'}^{\delta\delta} + C_\ell^{T\delta} C_{\ell'}^{T\delta} \sim C_\ell^{TT, \text{kSZ}} C_\ell^{\delta\delta}$.

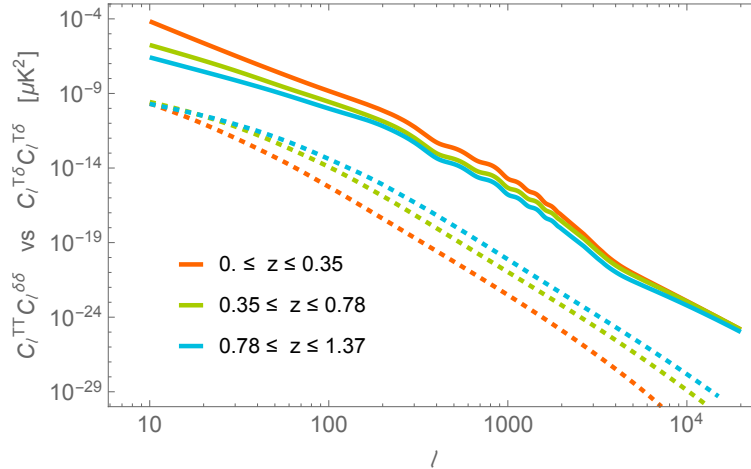


FIG. 7: The solid curves are $C_\ell^{TT} C_\ell^{\delta\delta}$ and the dashed curves are $C_\ell^{T\delta} C_\ell^{T\delta}$, which represent the two terms in square brackets in Eq. (71) evaluated at the same ℓ . The redshift ranges indicate sample redshift bins (see Fig. 4).

The final result for the cosmic variance limited noise estimate is

$$\sqrt{\langle \tilde{b}_L(\bar{\chi}_e)^2 \rangle} = \sqrt{\sum_{\ell, \ell'} [C_\ell^{TT, \text{kSZ}} C_{\ell'}^{\delta\delta}(\bar{\chi}_e)] \frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)} \left| C_{000}^{\ell\ell'L} \right|^2}. \quad (72)$$

where we have restored an explicit dependence on the redshift bin $\bar{\chi}_e$.

VI. COSMIC VARIANCE LIMITED SIGNAL TO NOISE

In this section, we assess the signal to noise for the power multipoles Eq. 48 in the cosmic variance limit using both a theoretical estimate and the simulations described in Sec. III. In each case, the signal to noise in each bin $\bar{\chi}_e$ is calculated as

$$\frac{S}{N}(\bar{\chi}_e) = \frac{b_{LM}(\bar{\chi}_e)}{\sqrt{\langle \tilde{b}_L(\bar{\chi}_e)^2 \rangle}} \quad (73)$$

where $b_{LM}(\bar{\chi}_e)$ is found using Eq. 48 and $\sqrt{\langle \tilde{b}_L(\bar{\chi}_e)^2 \rangle}$ is given by Eq. 72.

A. RMS Estimate

A simple estimate of the signal is obtained by approximating $a_{LM}^v(\chi_e) \sim \sqrt{C_L^v(\chi_e)}$ in Eq. 48, where $C_L^v(\chi_e)$ is the power spectrum associated with the large-scale velocity, given by

$$C_L^v(\chi_e) = \int_0^{k_{\max}} \frac{k^2 dk}{(2\pi)^3} P_\Psi(k) |\Delta_L^v(k, \chi_e)|^2 \quad (74)$$

with $\Delta_L^v(k, \chi_e)$ given by Eq. (32). This necessarily yields an overestimate of the signal, since in reality $a_{L0}^v(\chi_e)$ will vary over the window functions leading to partial cancellation, while here we are assuming that it always takes its (positive definite) RMS value. By comparing with simulations in the following subsection, we show that this approximation gives a good estimate in the limit of thin window functions $W(\chi_e, \bar{\chi}_e)$ (in the context of this paper, this is equivalent to the limit of many redshift bins).

Before proceeding, we can assess which scales form the dominant contribution to Eq. 74. The upper limit of integration in Eq. 74, k_{\max} , corresponds to the smallest scale, $\lambda_{\min} = 2\pi/k_{\max}$, that contributes to the signal. Formally, $k_{\max} \rightarrow \infty$, but we can adjust the cutoff to include only the long modes discussed above. In Fig. 8 we show C_L^v at $z = 1$ as a function of k_{\max} . Vertical lines indicate $\lambda_{\min} = 10, 10^2, 10^3$ Mpc. Here, we see that for a number of power multipoles, the relevant signal is obtained almost entirely from scales $\sim 10^2 - 10^3$ Mpc.

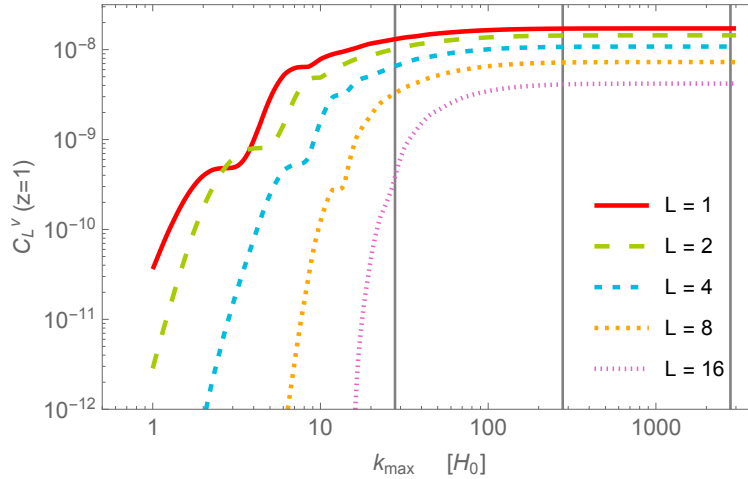


FIG. 8: The effective velocity contribution to the signal described in Eq. 74 versus k_{\max} in units of H_0 , which refers to the upper bound on the integral. Notice that the contribution mainly comes from large scales (small k) for low L . The vertical lines indicate scales equal to 1 Gpc, 100 Mpc, and 10 Mpc from left to right.

Putting everything together, applying the Limber approximation [59], and choosing $M = 0$ under the assumption that all other M will statistically be the same, the final expression for the signal becomes

$$b_{L0}(\bar{\chi}_e) \simeq \sum_{\ell} \frac{2\ell+1}{4\pi} \int \frac{dk}{\ell+1/2} \sigma_T a(\chi) \bar{n}_e(\chi) \sqrt{C_L^v(\chi)} W(\chi, \bar{\chi}_e) P_\delta(k, \chi) \Big|_{\chi \rightarrow (\ell+1/2)/k}. \quad (75)$$

We compute the signal Eq. 75 and noise Eq. 72 for the $N_{\text{bin}} = 6, 12$, and 24 top-hat bin configurations with redshift ranges summarized in Fig. 4. We show the results for the 6-bin configuration in Fig. 9. Each plot has four curves. The solid red curve is the RMS signal with $k_{\text{max}} \rightarrow \infty$, the dotted orange line is the cosmic variance limited noise, the short dashed blue line is the RMS signal computed for $k_{\text{max}} = 278 H_0$ ($\lambda_{\text{min}} = 100$ Mpc), and the long dashed green curve is the signal computed for $k_{\text{max}} = 2780 H_0$ ($\lambda_{\text{min}} = 10$ Mpc). Comparing the three signal curves we see that except in the lowest redshift bin at high L , the signal is composed primarily of long wavelength modes ($\lambda > 100$ Mpc), as expected. The amplitude of the signal varies by roughly an order of magnitude between the lowest and highest redshift bin, and is strongest at low L and low redshift. Both the signal and noise are relatively flat over the plotted range in L . Most importantly though, the signal is 2-3 orders of magnitude larger than the noise!

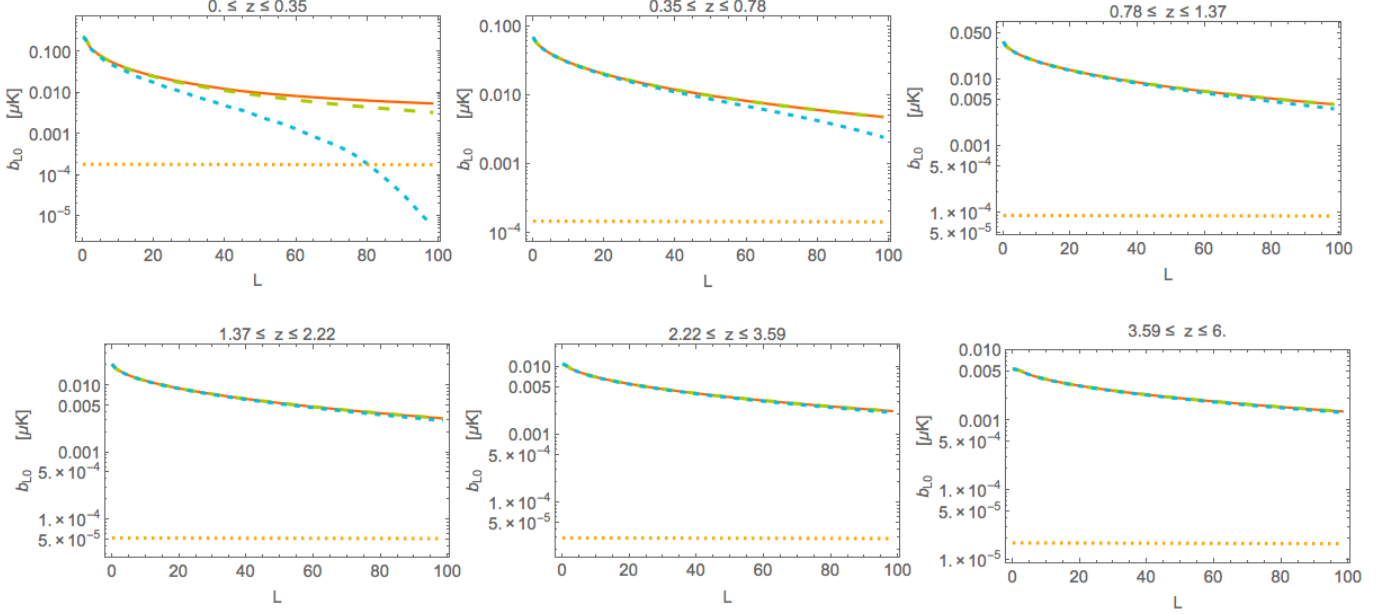


FIG. 9: The signal (75) is shown in six redshift bins, calculated for three different scales: $k_{\text{max}} = 278 H_0$ ($\lambda_{\text{min}} = 100$ Mpc) [blue, short dash], $k_{\text{max}} = 2780 H_0$ ($\lambda_{\text{min}} = 10$ Mpc) [green, long dash], and $k_{\text{max}} = \infty$ ($\lambda_{\text{min}} = 0$) [red, solid]. The dotted orange line is the noise estimate $\sqrt{\langle \tilde{b}_L^2 \rangle}$ (Eq. (72)), which falls well below the estimated signal for this configuration.

To compare the result for the 12 and 24 bin configurations, in Fig. 10 we show the signal to noise Eq. 73 at $L = 1$ and $L = 50$ as a function of comoving distance. Increasing the number of bins by a factor of 2 results in a decrease in the signal-to-noise by a factor of $\sim \sqrt{2}$. This is true for all L . We therefore conclude that a signal can in principle be measured in the cosmic variance limit at high signal to noise for a large number of redshift bins N_{bins} at a variety of scales L .

B. Comparing with the signal from simulations

The above calculation is an over-estimate of the signal, as it assumes the velocity field is positive definite and given by its RMS value. In particular, it does not account for partial cancellations along the line of sight. To take this into account, we can compute the signal from Eq. 48 using the effective velocity field computed from the simulations described in Sec. III. Using an interpolating function for $a_{\ell m}^v(\chi_e)$ constructed from the v_{eff} map in each of the 50 redshift bins, we compute Eq. 48 directly for 100 realizations at a resolution of $k_{\text{max}} \sim 57.4 H_0$ for the 6 and 12 bin configurations. Below we only present results for b_{L0} ; other values of M have identical statistical properties.

Figures 11 and 12 shows the signal to noise computed using simulations, in comparison to the one estimated from theory with Eq. 74 computed using the integration limit $k_{\text{max}} \sim 57.4 H_0$ corresponding to the simulation resolution. We plot $L < 10$, which is accurately captured for the resolution we consider (see Fig. 8). Notice that the solid curves, showing the predicted signal based on our theory calculation, are always higher than the average signal from the realizations. This is due to a difference in the order of operations. In our estimation in (75) we have averaged over $a_{L0}(\chi_e)$ prior to integrating over χ_e , whereas the signal computed from the realizations integrates over χ_e first

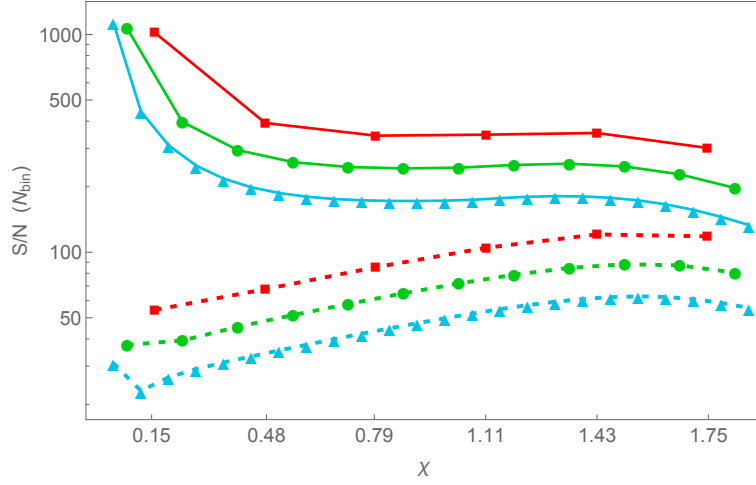


FIG. 10: The bullet points (square, circle, triangle) represent the signal-to-noise (Eq. (73)) in each bin for $N_{\text{bin}} = (6, 12, 24)$, plotted against χ at the midpoint of the bin. The solid top curves are for $L = 1$ and the bottom dashed curves are for $L = 80$. The signal and noise were computed using Eq. (75) and (72) with $k_{\text{max}} \rightarrow \infty$ in Eq. (74).

and then averages. Since $a_{L0}(\chi_e)$ is an oscillating function that takes positive and negative values, there can be cancellation upon integration over χ_e . This cancellation can be minimized by using smaller redshift bins, which results in a smaller range of integration and a lesser chance for cancellation. This can be noticed empirically as the agreement between realizations and theory is better for the 12 bin configuration than the 6 bin. In summary, the realizations approach the theory estimation for more bins as a result of having less variation in a_{LM} over the bin.

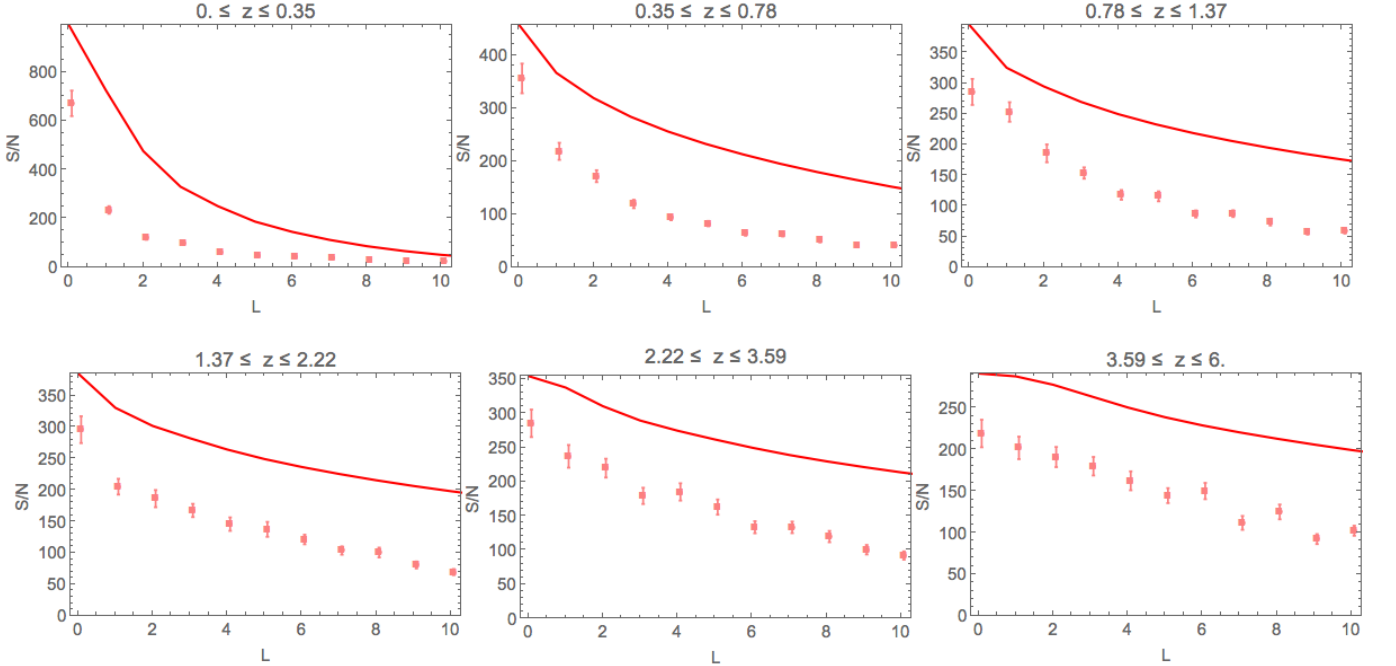


FIG. 11: The signal-to-noise (Eq. (73)) computed using simulations, in comparison to the RMS estimate (solid curves), in 6 redshift bins (see Fig. 4). The points represent the standard deviation of the 100 realizations, and the error bars denote the standard error of the standard deviation. These simulations have a resolution of $\lambda_{\text{min}} \sim 484$ Mpc.

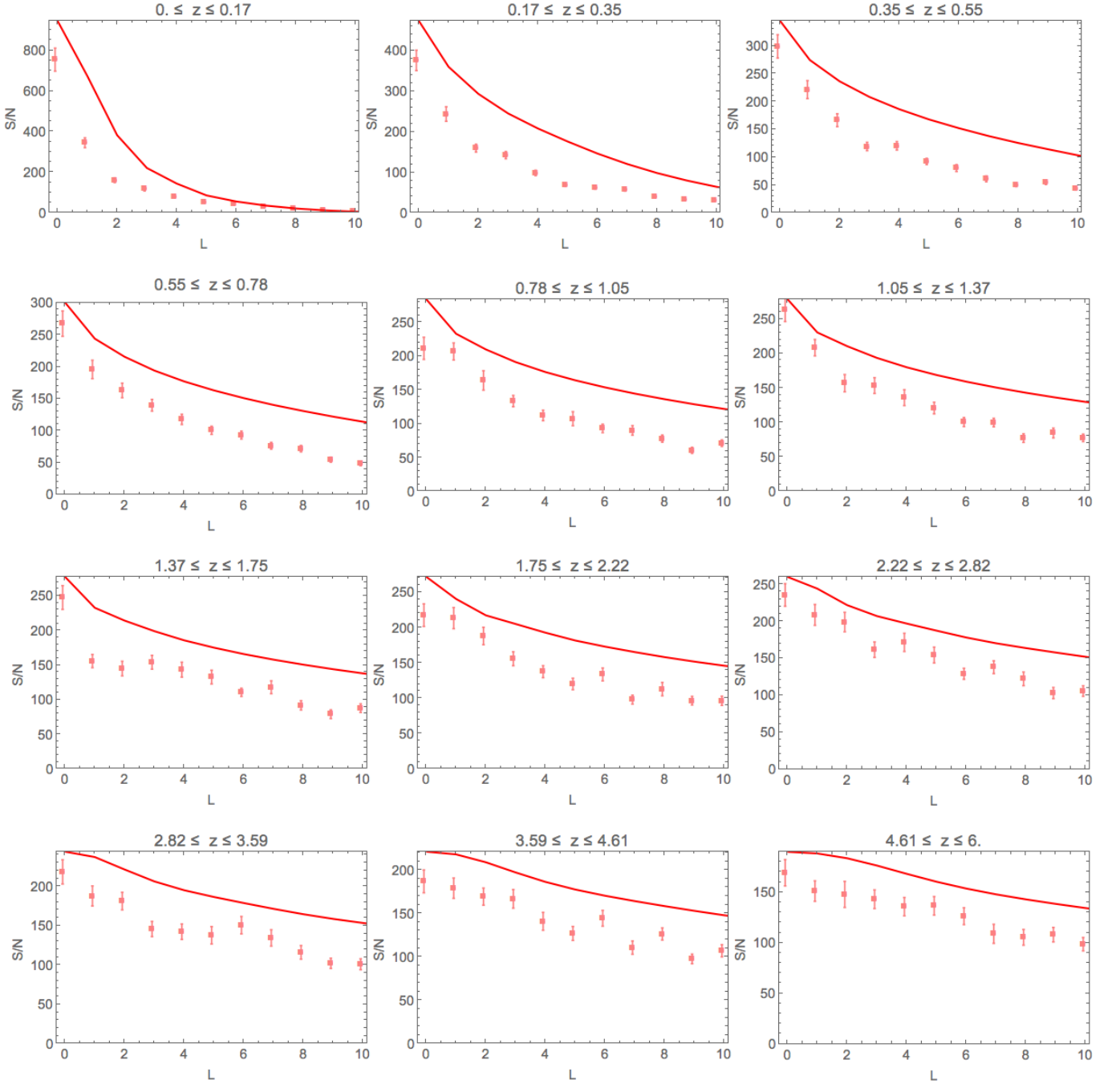


FIG. 12: The same as Figure 11 but for 12 redshift bins.

VII. MODE COUNTING

With an understanding of this signal in hand, we now want to estimate how many modes one could conceivably measure in the cosmic variance limit. Let's consider scales $\lambda > 100$ Mpc, of order the BAO scale. This corresponds to $k_{\max} = 278 H_0$ and $L_{\max} \sim \pi/\theta \sim \pi\chi_{\text{dec}}/\lambda_{\min} = k_{\max}\chi_{\text{dec}}/2 \sim 437$. On this scale and larger, the primary CMB contains $\sum_L^{437} (2L+1) = 191843$ modes.

For the kSZ effect considered here, the sum over $2L+1$ is performed in each bin, therefore one might naively guess that it is possible to get N_{bin} times more modes than the primary CMB. However, a more careful estimate needs to be done because the value of L_{\max} varies in each bin depending on the size of the signal. Consider the signal-to-noise,

N_{bin}	L_{max}	$\sum_{\text{bin}} \sum_L^{L_{\text{max}}} (2L + 1)$
6	(5, 32, 55, 78, 100, 123)	36073
12	(1, 16, 27, 39, 50, 61, 72, 84, 95, 106, 117, 130)	73414
24	(1, 8, 14, 19, 25, 31, 36, 42, 48, 53, 59, 64, 70, 75, 81, 87, 92, 98, 103, 109, 115, 119, 129, 131)	147577

TABLE I: Number of modes

described by the ratio of (75) and (72), for the scales considered here (see Eq. (74) and use $k_{\text{max}} = 278 H_0$). For N_{bin} bins, in each bin, the value of L_{max} up to which modes can be measured is found by ensuring that

1. For $L < L_{\text{max}}$, the signal-to-noise is bigger than 1.
2. For $L < L_{\text{max}}$, the signal is dominated by modes larger than 100 Mpc. More precisely, $b_{L0}^{k_{\text{max}}} > 0.95 b_{L0}^{\infty}$ for $k_{\text{max}} = 278 H_0$.³

Table I shows the values of L_{max} computed in each bin for the 6, 12 and 24 bin configurations. In every case, it was a failure of criteria (2.) that determined L_{max} , as the signal-to-noise is always much bigger than 1 for this range in L . Notice that by doubling the bin size, we approximately double the number of modes. This allows us to extrapolate our results from the three bin configurations.

Figure 13 shows how the number of modes increases with N_{bin} based on our estimates using 6, 12 and 24 bins. Extrapolating the data points to higher values of N_{bin} shows that at least 31 bins are needed to match the number of modes in the primary CMB. The same increasing trend should continue until $N_{\text{bin}} \sim 50$ (producing 309656 modes), at which point we estimate that the signal-to-noise will drop below 1 in the high redshift bins, thus failing to satisfy criteria (1.), and causing a less rapid increase in the number of modes as a function of N_{bin} . Therefore, this method can theoretically achieve more modes than the primary CMB. However, note that within the observable Universe, on scales $\lambda > 100$ Mpc, there are approximately $(\chi_{\text{dec}} k_{\text{max}} / (2\pi))^3 \sim 2.7 \times 10^6$ total modes. Thus, while remaining competitive with the primary CMB, the proposed method still falls about an order of magnitude short of the *total* number of possible modes. In addition, because they provide at least partially independent constraints, combining the information from the primary CMB and the information from kSZ tomography can in principle constrain a larger number of modes than either individually.

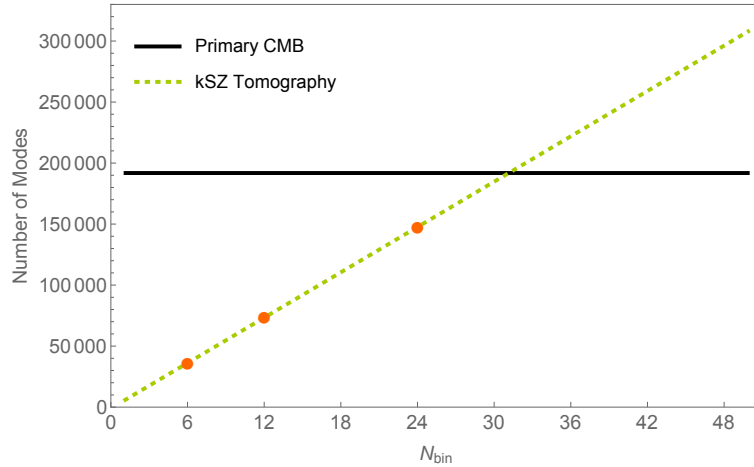


FIG. 13: The solid bullet points show the number of modes that can possibly be measured using $N_{\text{bin}} = 6, 12$, and 24. Extrapolating these data points (dashed line) indicates that we need at least 31 bins to match the number of modes in the primary CMB at $\ell \leq 437$ (solid line).

³ The 95% criteria is arbitrary.

VIII. DISCUSSION AND CONCLUSIONS

In this paper, we have assessed the ability of kSZ tomography to yield information about the long-wavelength Universe in the cosmic variance limit. The signal of interest is a power asymmetry in the direct cross correlation of the kSZ contribution to the CMB temperature and the electron density binned at various redshifts. We quantify this signal in terms of power multipoles, and compared it with the amplitude of “accidental” power asymmetry due to the statistically homogeneous component of the kSZ contribution to the CMB temperature (the primary source of cosmic variance in this context). The results are promising in this highly optimistic scenario, yielding a signal to noise greater than unity over a large range of power multipoles in a large number of redshift bins.

Although we have established that there is in principle a signal to detect, there is a significant amount of work that must be done to assess what can be done in practice. First, a proper forecast including CMB instrumental noise, beams, point spread functions, shot noise, imperfect foreground subtraction, and redshift errors should be done to determine what is required to reach the necessary threshold in sensitivity, accuracy, and resolution. Next, there are important systematic errors and other potential sources of power asymmetry that should be investigated including, but not limited to: relativistic aberration of the CMB [63] and large scale structure [64, 65], clustering [66], electron bias in the nonlinear regime (e.g. [67]), non-Gaussian aspects of CMB lensing [68], redshift space distortions [69], asymmetric scan strategies (e.g. [70]) and incomplete sky coverage, incomplete LSS surveys, asymmetric beams and point spread functions, and more realistic window functions. Furthermore, the signal may be boosted in the presence of some types of primordial non-Gaussianity [71]. We leave a more careful investigation of these and other important aspects to future work.

If it is indeed possible to approach the cosmic variance limited scenario we have described in this paper, what would we stand to learn? Because kSZ tomography is probing different portions of the surface of last scattering than the primary CMB, the constraining power for various early Universe scenarios involving extra sources of inhomogeneity can be extraordinary. For example, in Ref. [45] it was shown that the constraints on parameters in a theory that predicts cosmic bubble collisions could improve by several orders of magnitude comparing kSZ tomography to existing and forecasted constraints from the primary CMB. One might expect constraints on theories of the large-scale CMB anomalies to be similarly impressive. Given a high enough fidelity measurement, it should also be possible to reconstruct the 3D large-scale gravitational potential throughout much of the observable Universe (in analogy with Ref. [72]). Such a reconstruction could be an important tool, for example in future studies of primordial non-Gaussianity. Performing this exercise would also significantly clarify precisely what new information, beyond that encoded in the primary CMB and CMB polarization, there is to gain from kSZ tomography. In any case, we stand to learn a great deal about cosmology in the coming era of precision measurements of CMB secondaries, with the kSZ effect playing a leading role. In this new era, it is important to understand the nuances of the kSZ effect, and to target new observables.

Acknowledgments

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Appendix A: The effective velocity in Fourier space

In this appendix, we derive an expression for the effective line of sight velocity v_{eff} in Fourier space. The Fourier transform of the primordial potential is defined as

$$\Psi_i(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} e^{i\Delta\chi \mathbf{k} \cdot \hat{\mathbf{n}}}, \quad (\text{A1})$$

where we have explicitly expanded the position $\mathbf{r} = \chi_e \hat{\mathbf{n}}_e + \Delta\chi \hat{\mathbf{n}}$.

1. Sachs-Wolfe

Using Eq. 7, the SW contribution to the effective velocity is related to the Fourier components of Ψ_i through

$$v_{\text{eff,SW}}(\hat{\mathbf{n}}_e, \chi_e) = \frac{3}{4\pi} \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) \int d^2\hat{\mathbf{n}} \Psi_i(\mathbf{r}) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) \quad (\text{A2})$$

$$= \frac{3}{4\pi} \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \int d^2\hat{\mathbf{n}} e^{i\Delta\chi \mathbf{k} \cdot \hat{\mathbf{n}}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A3})$$

We can work on the second integral by expanding the exponential in terms of Legendre polynomials and spherical Bessel functions:

$$e^{i\Delta\chi \mathbf{k} \cdot \hat{\mathbf{n}}} = \sum_{\ell'} i^{\ell'} (2\ell' + 1) j_{\ell'}(k\Delta\chi) \mathcal{P}_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) \quad (\text{A4})$$

Substituting and applying the identity,

$$\int d^2\hat{\mathbf{b}} \mathcal{P}_{\ell'}(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) \mathcal{P}_\ell(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) = \frac{4\pi}{2\ell + 1} \mathcal{P}_\ell(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}}) \delta_{\ell\ell'} , \quad (\text{A5})$$

results in

$$\int d^2\hat{\mathbf{n}} e^{i\Delta\chi \mathbf{k} \cdot \hat{\mathbf{n}}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) = 4\pi i j_1(k\Delta\chi) \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A6})$$

Putting this result back into Eq. A2 gives

$$v_{\text{eff,SW}}(\hat{\mathbf{n}}_e, \chi_e) = 3i \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}_i(\mathbf{k}) j_1(k\Delta\chi_{\text{dec}}) \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (\text{A7})$$

We therefore see that $v_{\text{eff,SW}}$ is simply a convolution of the potential field evaluated on the intersection of the electron's past light cone and the time of decoupling.

2. Doppler

Using Eq. 10, the Doppler contribution to v_{eff} is

$$v_{\text{eff,Doppler}}(\hat{\mathbf{n}}_e, \chi_e) = \frac{3}{4\pi} D_v(\chi_{\text{dec}}) \int d^2\hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla \Psi_i(\mathbf{r}_{\text{dec}})) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) \quad (\text{A8})$$

$$- \frac{3}{4\pi} D_v(z_e) \int d^2\hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla \Psi_i(\mathbf{r}_e)) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) \quad (\text{A9})$$

We'll start with the first integral, then do the second. Going to Fourier space gives

$$\frac{3}{4\pi} D_v(\chi_{\text{dec}}) \int \frac{d^3k}{(2\pi)^3} ik \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \int d^2\hat{\mathbf{n}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) e^{i\Delta\chi_{\text{dec}} \mathbf{k} \cdot \hat{\mathbf{n}}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A10})$$

Expanding the exponent in Legendre polynomials results in

$$\frac{3}{4\pi} D_v(\chi_{\text{dec}}) \int \frac{d^3k}{(2\pi)^3} ik \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \sum_{\ell'} i^{\ell'} (2\ell' + 1) j_{\ell'}(k\Delta\chi_{\text{dec}}) \int d^2\hat{\mathbf{n}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \mathcal{P}_{\ell'}(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A11})$$

The integral over three Legendre polynomials can be evaluated by expanding in spherical harmonics and using the Wigner 3j symbols. The result is

$$\int d^2\hat{\mathbf{n}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \mathcal{P}_{\ell'}(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) = \frac{4\pi}{3} \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) \delta_{\ell'0} + \frac{8\pi}{15} \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) \delta_{\ell'2} , \quad (\text{A12})$$

and upon substitution, we obtain for the first integral:

$$iD_v(\chi_{\text{dec}}) \int \frac{d^3k}{(2\pi)^3} k \tilde{\Psi}_i(\mathbf{k}) [(j_0(k\Delta\chi_{\text{dec}}) - 2j_2(k\Delta\chi_{\text{dec}}))] \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (\text{A13})$$

Moving to the second integral in Eq. A8, we have

$$- \frac{3}{4\pi} D_v(\chi_e) \int \frac{d^3 k}{(2\pi)^3} i k \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \int d^2 \hat{\mathbf{n}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A14})$$

The integral over angles can be evaluated using the identity Eq. A5 to obtain

$$- i D_v(\chi_e) \int \frac{d^3 k}{(2\pi)^3} k \tilde{\Psi}_i(\mathbf{k}) \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (\text{A15})$$

Assembling the various pieces, the Doppler contribution becomes:

$$v_{\text{eff,Doppler}}(\hat{\mathbf{n}}_e, \chi_e) = i \int \frac{d^3 k}{(2\pi)^3} k \tilde{\Psi}_i(\mathbf{k}) [D_v(\chi_{\text{dec}}) j_0(k\Delta\chi_{\text{dec}}) - 2D_v(\chi_{\text{dec}}) j_2(k\Delta\chi_{\text{dec}}) - D_v(\chi_e)] \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (\text{A16})$$

Note that the last term in square brackets contains no spherical Bessel function. Therefore, the Doppler component receives contributions from all scales, unlike the Sachs-Wolfe term.

3. Integrated Sachs-Wolfe

Using Eq. 14, the ISW contribution to the effective velocity is

$$v_{\text{eff,ISW}}(\hat{\mathbf{n}}_e, \chi_e) = \frac{3}{4\pi} \int d^2 \hat{\mathbf{n}} \left(2 \int_{a_{\text{dec}}}^{a_e} \frac{dD_\Psi}{da} \Psi_i(\mathbf{r}(a)) da \right) \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) . \quad (\text{A17})$$

Going to Fourier space,

$$v_{\text{eff,ISW}}(\hat{\mathbf{n}}_e, \chi_e) = \frac{3}{2\pi} \int_{a_{\text{dec}}}^{a_e} da \frac{dD_\Psi}{da} \int \frac{d^3 k}{(2\pi)^3} \tilde{\Psi}_i(\mathbf{k}) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} \int d^2 \hat{\mathbf{n}} e^{i\Delta\chi(a) \mathbf{k} \cdot \hat{\mathbf{n}}} \mathcal{P}_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_e) , \quad (\text{A18})$$

and applying the identity Eq. A6, we obtain

$$v_{\text{eff,ISW}}(\hat{\mathbf{n}}_e, \chi_e) = 6i \int \frac{d^3 k}{(2\pi)^3} \tilde{\Psi}_i(\mathbf{k}) \left[\int_{a_{\text{dec}}}^{a_e} da \frac{dD_\Psi}{da} j_1(k\Delta\chi(a)) \right] \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} . \quad (\text{A19})$$

Just as for the SW contribution, the ISW contribution is mainly sensitive to potential fluctuations on large scales.

4. Effective velocity

We can now assemble Eq. A7, Eq. A16, and Eq. A19 into an expression for the total effective velocity. Before doing so, it must be noted that the expression Eq. A16 for the Doppler kernel is only valid in the small- k limit, so the linear growth with k eventually gets cutoff. To fix this, we can incorporate the transfer function $T(k)$ by simply replacing $\tilde{\Psi}_i(\mathbf{k}) \rightarrow T(k) \tilde{\Psi}_i(\mathbf{k})$. We will employ the BBKS fitting function:

$$T(k) = \frac{\ln[1 + 0.171x]}{0.171x} [1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.49x)^4]^{-0.25} , \quad (\text{A20})$$

where $x = k/k_{\text{eq}}$ with $k_{\text{eq}} = a_{\text{eq}} H(a_{\text{eq}}) = \sqrt{2/a_{\text{eq}}} H_0 \simeq 82.5 H_0$. Putting together all three components, (A7), (A16), (A19), gives the expression for the effective velocity:

$$v_{\text{eff}}(\hat{\mathbf{n}}_e, \chi_e) = i \int \frac{d^3 k}{(2\pi)^3} T(k) \tilde{\Psi}_i(\mathbf{k}) [\mathcal{K}_D(k, \chi_e) + \mathcal{K}_{\text{SW}}(k, \chi_e) + \mathcal{K}_{\text{ISW}}(k, \chi_e)] \mathcal{P}_1(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}_e) e^{i\chi_e \mathbf{k} \cdot \hat{\mathbf{n}}_e} , \quad (\text{A21})$$

where the kernels are defined as

$$\mathcal{K}_D(k, \chi_e) \equiv k D_v(\chi_{\text{dec}}) j_0(k\Delta\chi_{\text{dec}}) - 2k D_v(\chi_{\text{dec}}) j_2(k\Delta\chi_{\text{dec}}) - k D_v(\chi_e) \quad (\text{A22})$$

$$\mathcal{K}_{\text{SW}}(k, \chi_e) \equiv 3 \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) j_1(k\Delta\chi_{\text{dec}}) \quad (\text{A23})$$

$$\mathcal{K}_{\text{ISW}}(k, \chi_e) \equiv 6 \int_{a_{\text{dec}}}^{a_e} da \frac{dD_\Psi}{da} j_1(k\Delta\chi(a)) \quad (\text{A24})$$

Appendix B: A pure gradient is pure gauge

In this appendix we explicitly demonstrate that a pure gradient in the Newtonian potential Ψ (or more generally, the curvature perturbation in an arbitrary gauge) can be removed through a special conformal transformation. More generally, we can remove the gradient of the Newtonian potential at a point, which we take to be the origin of Cartesian coordinates. The Newtonian potential appears in a conformal factor in front of the spatial metric:

$$ds_3^2 = (1 - 2\Psi(\mathbf{x}))\delta_{ij}dx^i dx^j . \quad (\text{B1})$$

Performing a special conformal transformation

$$x^i = \frac{x'^i - b^i x'_i x'^i}{1 - 2b_i x'^i + (b_i b^i)(x'_i x'^i)} , \quad (\text{B2})$$

takes the spatial metric to

$$\delta_{ij}dx^i dx^j = \frac{\delta_{ij}}{[1 - 2b_i x'^i + (b_i b^i)(x'_i x'^i)]^2} dx'^i dx'^j , \quad (\text{B3})$$

where b_i are free constants.

If we imagine there was a pure gradient in the Newtonian potential,

$$\Psi(\mathbf{x}) = A_i x^i , \quad (\text{B4})$$

we can write

$$(1 - 2\Psi(\mathbf{x}))\delta_{ij}dx^i dx^j = \left(1 - 2A_i \frac{x'^i - b^i x'_i x'^i}{1 - 2b_i x'^i + (b_i b^i)(x'_i x'^i)}\right) \frac{\delta_{ij}}{[1 - 2b_i x'^i + (b_i b^i)(x'_i x'^i)]^2} dx'^i dx'^j . \quad (\text{B5})$$

For $b_i \ll 1$, expanding to first order, we have:

$$(1 - 2\Psi(\mathbf{x}))\delta_{ij}dx^i dx^j \simeq (1 - 2(A_i - 2b_i)x'^i + \mathcal{O}(b^2)) \delta_{ij}dx'^i dx'^j . \quad (\text{B6})$$

Therefore, with the choice,

$$b_i = A^i/2 , \quad (\text{B7})$$

we have $\Psi(\mathbf{x}') = 0$, and therefore no gradient in the primed coordinate system. Note also that the choice for b_i justifies neglecting the terms of higher order in b , at least in linear perturbation theory.

More generally, we could imagine performing a Taylor series expansion of $\Psi(\mathbf{x})$ about a point, taken here to be $\mathbf{x} = 0$:

$$\Psi(\mathbf{x}) \simeq \Psi(0) + \partial_i \Psi(0)x^i + \dots \quad (\text{B8})$$

If we perform a special conformal transformation with

$$b_i = \partial_i \Psi(0)/2 , \quad (\text{B9})$$

then in the primed coordinates we have

$$\partial_i \Psi(0)x^i = \frac{1}{2} - \frac{1 - 3\partial_i \Psi(0)x'^i + (\partial_i \Psi(0)\partial^i \Psi(0))(x'_i x'^i)/4 + (\partial^i \Psi(0)x'^i)^2}{2[1 - \partial_i \Psi(0)x'^i + (\partial_i \Psi(0)\partial^i \Psi(0))(x'_i x'^i)/4]} . \quad (\text{B10})$$

To lowest order in $\partial_i \Psi(0)$, this is

$$\partial_i \Psi(0)x^i \simeq (\partial_i \Psi(0)x'^i)^2 + (\partial_i \Psi(0)\partial^i \Psi(0))(x'_i x'^i)/4 . \quad (\text{B11})$$

Therefore, in the primed coordinate system, the Taylor series expansion of $\Psi(\mathbf{x}')$ is

$$\Psi(\mathbf{x}') \simeq \Psi(0) + \mathcal{O}(x'^2) + \dots \quad (\text{B12})$$

with no linear term as advertised. Therefore, the special conformal transformation can be used to eliminate the derivative of Ψ at a point. This comes at the price of altering the higher-order terms in the Taylor series expansion.

Appendix C: Cancellation of the kernel contributions as $k \rightarrow 0$ for Λ CDM without radiation

In this Appendix we show the exact cancellation of the three contributions to the kernel of the effective velocity given in (17)-(19) for the largest scales in a universe with only matter and Λ . In this case, $y = a/a_{\text{eq}} \rightarrow \infty$, so we can approximate $D_\Psi(a)$ using

$$D_\Psi(a) \equiv \frac{\Psi_{\text{SH}}(a)}{\Psi_{\text{SH},i}} = \frac{9}{10} \left[\frac{5}{2} \Omega_m \frac{E(a)}{a} G(a) \right], \quad (\text{C1})$$

where $G(a) \equiv \int_0^a da' \left[E(a') a' \right]^{-3}$ and $E(a) = \sqrt{\Omega_m a^{-3} + \Omega_\Lambda}$. Further, the distance along the electron's past light cone to redshift $z = 1/a - 1$, normalized by H_0 is given by

$$\Delta\chi(a) = - \int_{a_e}^a da' \frac{1}{E(a') a'^2}. \quad (\text{C2})$$

Let's begin with the simple Sachs-Wolfe term. Expanding to linear order in k , we obtain

$$\mathcal{K}_{\text{SW}} = 3 \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) j_1(k\Delta\chi_{\text{dec}}) \quad (\text{C3})$$

$$= 3 \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) \frac{k\Delta\chi_{\text{dec}}}{3} + \mathcal{O}(k^3) \quad (\text{C4})$$

$$= \left(2D_\Psi(\chi_{\text{dec}}) - \frac{3}{2} \right) k\Delta\chi_{\text{dec}} + \mathcal{O}(k^3). \quad (\text{C5})$$

Expanding the Bessel functions, the Doppler piece will only have two terms at linear order:

$$\mathcal{K}_{\text{D}} = -kD_v(a_e) + kD_v(a_{\text{dec}}) + \mathcal{O}(k^3) \quad (\text{C6})$$

We can simplify D_v as follows

$$D_v(a) = \frac{2a^2 H(a)}{\Omega_m} \frac{y}{4 + 3y} \left[D_\Psi(a) + \frac{dD_\Psi(a)}{d \ln a} \right] \quad (\text{C7})$$

$$= \frac{2a^2 E(a)}{\Omega_m} \frac{1}{3} \left[D_\Psi(a) + a \frac{dD_\Psi(a)}{da} \right] \quad (\text{C8})$$

$$= \frac{2a^2 E(a)}{3\Omega_m} \left[-\frac{3\Omega_m}{2E^2(a)a^3} \left(D_\Psi(a) - \frac{3}{2} \right) \right] \quad (\text{C9})$$

$$= -\frac{1}{aE(a)} \left(D_\Psi(a) - \frac{3}{2} \right). \quad (\text{C10})$$

This allows us to write the Doppler piece of the kernel as

$$\mathcal{K}_{\text{D}} = \frac{k}{a_e E(a_e)} \left(D_\Psi(a_e) - \frac{3}{2} \right) - \frac{k}{a_{\text{dec}} E(a_{\text{dec}})} \left(D_\Psi(a_{\text{dec}}) - \frac{3}{2} \right) + \mathcal{O}(k^3) \quad (\text{C11})$$

$$= \frac{3k}{2} \left(\frac{1}{a_{\text{dec}} E(a_{\text{dec}})} - \frac{1}{a_e E(a_e)} \right) + \frac{9k}{4} \Omega_m \left(\frac{G(a_e)}{a_e^2} - \frac{G(a_{\text{dec}})}{a_{\text{dec}}^2} \right) + \mathcal{O}(k^3). \quad (\text{C12})$$

The ISW term can be shown to exactly cancel the above two contributions. We start by expanding $j_1(k\Delta\chi(a)) \sim (k\Delta\chi(a))/3$ and integrating by parts:

$$\mathcal{K}_{\text{ISW}} = 6 \int_{a_{\text{dec}}}^{a_e} da \frac{dD_\Psi(a)}{da} j_1(k\Delta\chi(a)) \quad (\text{C13})$$

$$= \left[6D_\Psi(a) \frac{k\Delta\chi(a)}{3} \right]_{a=a_{\text{dec}}}^{a=a_e} - 6 \int_{a_{\text{dec}}}^{a_e} da \frac{kD_\Psi(a)}{3} \frac{d\Delta\chi(a)}{da} + \mathcal{O}(k^3) \quad (\text{C14})$$

$$= -2D_\Psi(a_{\text{dec}})k\Delta\chi_{\text{dec}} + 2k \int_{a_{\text{dec}}}^{a_e} da \frac{D_\Psi(a)}{a^2 E(a)} + \mathcal{O}(k^3) \quad (\text{C15})$$

$$= -2D_\Psi(a_{\text{dec}})k\Delta\chi_{\text{dec}} + \frac{9k}{2} \Omega_m \int_{a_{\text{dec}}}^{a_e} da \frac{G(a)}{a^3} + \mathcal{O}(k^3). \quad (\text{C16})$$

The integral in the second term becomes

$$\int_{a_{\text{dec}}}^{a_e} da \frac{G(a)}{a^3} = \left[\frac{-G(a)}{2a^2} \right]_{a=a_{\text{dec}}}^{a=a_e} + \int_{a_{\text{dec}}}^{a_e} \frac{da}{2a^5 E^3(a)} \quad (\text{C17})$$

$$= -\frac{1}{2} \left(\frac{G(a_e)}{a_e^2} - \frac{G(a_{\text{dec}})}{a_{\text{dec}}^2} \right) + \frac{1}{3\Omega_m} \int_{a_{\text{dec}}}^{a_e} \frac{da}{a} \frac{d}{da} \left[\frac{1}{E(a)} \right] \quad (\text{C18})$$

$$= -\frac{1}{2} \left(\frac{G(a_e)}{a_e^2} - \frac{G(a_{\text{dec}})}{a_{\text{dec}}^2} \right) + \frac{1}{3\Omega_m} \left(\left[\frac{1}{aE(a)} \right]_{a=a_{\text{dec}}}^{a=a_e} + \int_{a_{\text{dec}}}^{a_e} \frac{da}{a^2 E(a)} \right) \quad (\text{C19})$$

$$= -\frac{1}{2} \left(\frac{G(a_e)}{a_e^2} - \frac{G(a_{\text{dec}})}{a_{\text{dec}}^2} \right) - \frac{1}{3\Omega_m} \left(\frac{1}{a_{\text{dec}} E(a_{\text{dec}})} - \frac{1}{a_e E(a_e)} + \Delta\chi_{\text{dec}} \right). \quad (\text{C20})$$

Inserting this into (C16) gives the final result for the ISW contribution

$$\mathcal{K}_{\text{ISW}} = -2D_\Psi(a_{\text{dec}})k\Delta\chi_{\text{dec}} - \frac{9k}{4}\Omega_m \left(\frac{G(a_e)}{a_e^2} - \frac{G(a_{\text{dec}})}{a_{\text{dec}}^2} \right) - \frac{3k}{2} \left(\frac{1}{a_{\text{dec}} E(a_{\text{dec}})} - \frac{1}{a_e E(a_e)} + \Delta\chi_{\text{dec}} \right) + \mathcal{O}(k^3) \quad (\text{C21})$$

It is now clear that adding equations (C5), (C12) and (C21) gives

$$\mathcal{K}_{\text{SW}} + \mathcal{K}_{\text{D}} + \mathcal{K}_{\text{ISW}} = 0 + \mathcal{O}(k^3). \quad (\text{C22})$$

When radiation is included, this cancellation still holds as shown numerically in Fig. 2.

Appendix D: Random Gaussian fields

Following Ref. [54], we generate realizations of the primordial potential $\Psi(\mathbf{x})$ in a four-step process. Given a spatial grid of size L^3 with N^3 positions $\mathbf{x}(\mathbf{m}) = L\mathbf{m}/N$, labeled by the integer triplet \mathbf{m} with components $m_i \in [0, N)$:

1. Define a field $\xi(\mathbf{m})$ that lives on the grid. Draw $\xi(\mathbf{m})$ at each \mathbf{m} from an independent Gaussian pdf with variance N^3 .
2. Fourier transform to get

$$\xi(\kappa) = N^{-3} \sum_{\mathbf{m}} \exp \left[-\frac{2\pi i}{N} \kappa \cdot \mathbf{m} \right] \quad (\text{D1})$$

where $\kappa \equiv kL/(2\pi)$ is the dimensionless wavenumber.

3. Multiply $\xi(\kappa)$ by

$$F(k) \equiv \left[\left(\frac{2\pi}{L} \right)^3 P_\Psi(k) \right]^{1/2} \quad (\text{D2})$$

where $P_\Psi(k)$ is the Gaussian primordial power spectrum of Λ CDM. Here, we use parameters for the amplitude and spectral index consistent with Planck [56] ($A_s = 2.2 \times 10^{-9}$, $n_s = 0.96$).

4. Inverse Fourier transform to obtain a random field with the correct correlation properties in real and Fourier space:

$$\Psi_i(\mathbf{m}) = \sum_{\kappa} F(k) \xi(\kappa) \exp \left[\frac{2\pi i}{N} \kappa \cdot \mathbf{m} \right] \quad (\text{D3})$$

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